12. $y'' + 2y' + 2y = \frac{1}{2}\delta(t - \frac{\pi}{6}) \qquad y(0)=0,\ y'(0)=1$

$$s^2Y - s\cancel{y(0)} - y'(0) + 2(sY - \cancel{y(0)}) + 2Y = \frac{1}{2}e^{-\pi/6 s}$$

$$(s^2 + 2s + 2)Y = 1 + \frac{1}{2}e^{-\pi/6 s}$$

$$Y = \frac{1}{(s+1)^2+1} + \frac{1}{2}e^{-\pi/6 s}\frac{1}{(s+1)^2+1}$$

$$y(t) = e^{-t}\sin t + \frac{1}{2}u_{\frac{\pi}{6}}(t)\cdot e^{-t(-\pi/6)}\sin(t - \pi/6)$$

13. $$F(s) = \frac{2s+5}{s^2+2s+10} = \frac{2s+5}{s^2+2s+1+9} = \frac{2s}{(s+1)^2+3^2} + \frac{5}{(s+1)^2+3^2}$$

$$f(t) = 2e^{-t}\cos 3t + \frac{5}{3}e^{-t}\sin 3t$$

14 $\begin{bmatrix} 2 & 1 \\ k & 3 \end{bmatrix} \qquad \begin{vmatrix} 2-\lambda & 1 \\ k & 3-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)(3-\lambda) - k = 0$

$$\lambda^2 - 5\lambda + 6 - k = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6-k)}}{2} = \frac{5 \pm \sqrt{25 - 24 + 4k}}{2}$$

$$= \frac{5 \pm \sqrt{1+4k}}{5} \qquad \sqrt{1+4k} > 5 \quad \text{and} \quad 1+4k \geq 0$$

$$1 + 4k > 25 \qquad k > -1/4$$

$$4k > 24$$

$$k > 6$$

15.

$$\vec{x}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-9-\lambda) + 36 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda+3)^2 = 0 \qquad \lambda = -3, -3$$

$$\left[\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right] \qquad \left[\begin{array}{cc|c} 2 & -6 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$2u_1 - 3u_2 = 1$

$2u_1 = 3u_2 + 1$

$u_2 = r \qquad u_1 = \frac{3}{2}u_2 + \frac{1}{2}$

$u_1 = \frac{3r+1}{2}$

$= \frac{3}{2}r + \frac{1}{2}$

$\cancel{\vec{u} = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} +}$

$$\vec{u} = r\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-3t}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}\right.$$

16. $\vec{x}' = \begin{bmatrix} 2 & 4 \\ -6 & -8 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 4 \\ -6 & -8-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-8-\lambda) + 24 = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda+2)(\lambda+4) = 0 \qquad \lambda = -2,\ \lambda = -4$$

$\underline{\lambda=-2}$ $\left[\begin{array}{cc|c} 4 & 4 & 0 \\ -6 & -6 & 0 \end{array}\right]$ $\vec{v}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda=-4}$ $\left[\begin{array}{cc|c} +6 & 4 & 0 \\ -6 & -4 & 0 \end{array}\right]$ $\vec{v}=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$$\vec{x}(t)=c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}+c_2 e^{-4t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}+c_2\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & -3 & -1 \end{array}\right]$ $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -5 & -3 \end{array}\right]$ $c_2 = 3/5$

$c_1 = 2-2\left(\frac{3}{5}\right)$

$= 2-\frac{6}{5}=\frac{4}{5}$

$$\vec{x}(t)=\frac{4}{5}e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}+\frac{3}{5}e^{-4t}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

17. $\vec{x}\,'=A\vec{x}$ $\lambda=3+2i \quad 3-2i$ $\vec{v}=\begin{bmatrix} -2 \\ 1-i \end{bmatrix}$

$$\vec{x}_1(t)=e^{3t}(\cos 2t+i\sin 2t)\begin{bmatrix} -2 \\ 1-i \end{bmatrix}$$

$$=e^{3t}\begin{bmatrix} -2\cos 2t-2i\sin 2t \\ \cos 2t+\sin 2t+i\sin 2t-i\cos 2t \end{bmatrix}$$

$$=e^{3t}\left\{\begin{bmatrix} -2\cos 2t \\ \cos 2t+\sin 2t \end{bmatrix}+i\begin{bmatrix} -2\sin 2t \\ \sin 2t-\cos 2t \end{bmatrix}\right\}$$

$$\vec{x}(t)=c_1 e^{3t}\begin{bmatrix} -2\cos 2t \\ \cos 2t+\sin 2t \end{bmatrix}+c_2 e^{3t}\begin{bmatrix} -2\sin 2t \\ \sin 2t-\cos 2t \end{bmatrix}$$

18. $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ 5e^t \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(-2-\lambda)+3=0$

$\lambda^2 \cancel{+} - 3 = 0$ $\quad \cancel{\lambda = \pm\sqrt{3}}$ $\quad \lambda = \pm 1$

$\underline{\lambda = 1}$ $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\lambda = -1}$ $\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\vec{x}_h = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{\Psi} = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix}$ $\quad \Psi \vec{u}' = \vec{g}$

$\left[\begin{array}{cc|c} e^t & e^{-t} & e^t \\ e^t & 3e^{-t} & 5e^t \end{array}\right]$ $\quad u_2' = e^{3t} \quad u_2 = \frac{1}{3}e^{3t}$

$\left[\begin{array}{cc|c} 1 & e^{-2t} & 1 \\ 0 & 2e^{-2t} & 4e^t \end{array}\right]$ $\quad u_1' = 1 - e^{-2t}e^{3t} = 1 - e^t$

$u_1 = t - e^t$

$\left[\begin{array}{cc|c} 1 & e^{-2t} & 1 \\ 0 & 1 & e^{3t} \end{array}\right]$

$$\vec{x} = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} t - e^t + c_1 \\ \frac{1}{3}e^{3t} + c_2 \end{bmatrix}$$

22. $y'' + \lambda y = 0 \qquad y'(0) = 0, \; y(2) = 0$

$\cancel{\lambda = 2k^2} \qquad \cancel{k = 1, 2, 3} \qquad \lambda > 0$

$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$y' = -\sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}\, x + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}\, x$

$0 = \sqrt{\lambda}\, c_2 \quad \rightarrow c_2 = 0$

$0 = c_1 \cos(\sqrt{\lambda} \cdot 2) \qquad c_1 \neq 0 \qquad \cos(2\sqrt{\lambda}) = 0$

$2\sqrt{\lambda} = \frac{\pi}{2} + n\pi$

$\sqrt{\lambda} = \frac{\pi}{4} + n\frac{\pi}{2}$

$\lambda = \left(\frac{\pi}{4} + \frac{n\pi}{2}\right)^2$

$\lambda_n = \left(\frac{\pi + 2n\pi}{4}\right)^2$

$y_n = \cos\left(\frac{\pi + 2n\pi}{4} x\right)$