Final exam: Thu. 8/3 8 AM WTHR 172

1. $x^2y'' + \beta y = 0$

find β such that $y \to 0$ as $x \to 0^+$

Euler eq. $x^2y'' + \alpha x y' + \beta y = 0$

Solution: $y = c_1 x^{r_1} + c_2 x^{r_2}$ if $r_1 \neq r_2$

$y = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x)$ $r = \lambda \pm i\mu$

$y = c_1 x^r + c_2 x^r \ln x$ if $r_1 = r_2 = r$

indicial eq. $r(r-1) + \alpha r + \beta = 0$

$r(r-1) + \beta = 0$

$r^2 - r + \beta = 0$

$$r = \frac{1 \pm \sqrt{1-4\beta}}{2}$$

$r > 0$ if real

or $\lambda > 0$ if $r = \lambda \pm i\mu$

to ensure $y \to 0$ as $x \to 0^+$

$\sqrt{1-4\beta} < 1$ $\quad$ ~~$-1<1-4\beta<1$~~ $\quad$ $\mathbb{R}$

$1-4\beta < 1$ $\quad$ ~~$-2<-4\beta<0$~~ $\quad$ ~~$1/2 > \beta > 0$~~ $\quad$ roots will be real and positive

$-4\beta < 0$ $\quad \boxed{\beta > 0}$ $\quad$ ~~$0<\beta<1/2$~~

$1-4\beta < 0$ $\quad$ complex, real part > 0

$4\beta > 1$ $\quad \boxed{\beta > 1/4}$

$1-4\beta = 0$ $\quad$ repeat, both positive $\quad \boxed{\beta = \frac{1}{4}}$

so we can have all β's such that ~~$\beta>0$~~, ~~$\beta>1/4$~~ $\beta > 0$
~~$0<\beta<1/2$~~

2. $\quad y'' - 2(x-1)y' + y = 0 \quad y(1) = -1,\ y'(1) = 0$

$x_0 = 1$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} a_n (n) (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} 2a_n (n)(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)(x-1)^n - \sum_{n=1}^{\infty} 2a_n (n)(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$n=0:$ $\quad 2a_2 + a_0 = 0 \qquad a_2 = -\frac{1}{2}a_0 \qquad y(1) = -1 \rightarrow a_0 = -1$

$$a_2 = \frac{1}{2}$$

$n \geq 1:$ $\quad a_{n+2}(n+2)(n+1) - 2a_n(n) + a_n = 0$

$$a_{n+2} = \frac{(2n-1)\,a_n}{(n+2)(n+1)} \qquad y'(1) = 0 \rightarrow a_1 = 0$$

even/odd split

$a_1 = a_3 = a_5 = \cdots = 0$ (all odd n's)

$n=2:$ $\quad a_4 = \frac{3a_2}{4 \cdot 3} = \frac{1}{4}a_2 = \frac{1}{8}$

$n=4:$ $\quad a_6 = \frac{7a_4}{6 \cdot 5} = \frac{7}{30}\left(\frac{1}{8}\right) = \frac{7}{240}$

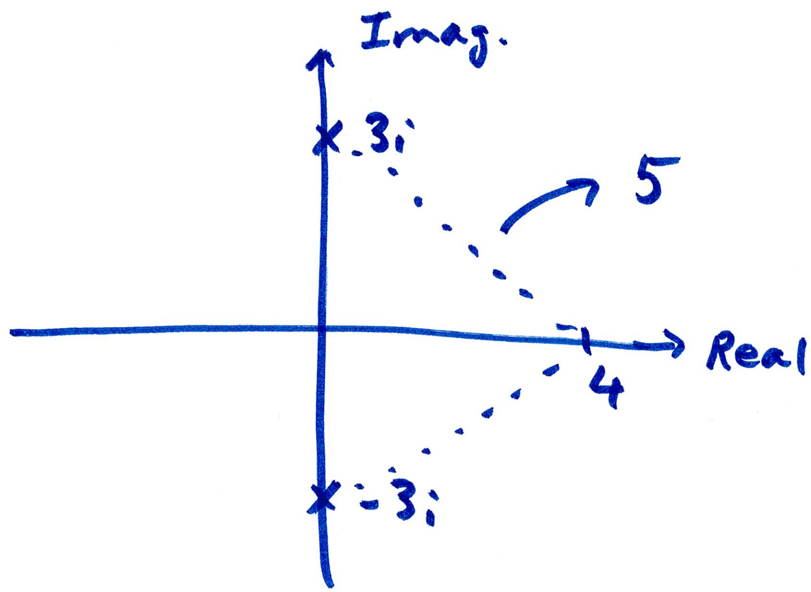
$$y(x) = -1 + \frac{1}{2}(x-1)^2 + \frac{1}{8}(x-1)^4 + \frac{7}{240}(x-1)^6 + \ldots$$

3. lower bound of radius of convergence

$$(x^2+9)y'' + xy' + 5y = 0 \qquad x_0 = 4$$

$$y'' + \frac{x}{x^2+9}y' + \frac{5}{x^2+9}y = 0$$

$$x^2+9=0 \rightarrow x = \pm 3i$$



$R \geq 5$

4, 5 series solution similar to # 2

skip for now

6. $x(x-1)y'' + 6x^2y' + 3y = 0$

$x_0 = 1$ is regular singular pt

find indicial eq.

hard way: $y = \sum_{n=0}^{\infty} a_n (x-1)^{r+n}$ plug into DE, find recurrence

easy way: $p_0 = \lim_{x \to 1} (x-1) \dfrac{6x^2}{x(x-1)} = 6$

$$q_0 = \lim_{x \to 1} (x-1)^2 \frac{3}{x(x-1)} = 0$$

indicial: $r(r-1) + p_0 r + q_0 = 0$

$$r(r-1) + 6r = 0$$

$$r^2 + 5r = 0$$

$$r = 0, \quad r = -5$$

7. Euler eq. (see # 1)

8. $$f(t) = \begin{cases} t^2 & 0 \leq < t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$$ find $\mathcal{L}\{f(t)\}$

$$f(t) = t^2 + u_1(t) \cdot (-2t + 2)$$

$$F(s) = \frac{2}{s^3} + e^{-s} \mathcal{L}\{-2(t+1) + 2\}$$

$$= \frac{2}{s^3} + e^{-s}\left(\frac{-2}{s^2}\right)$$

9. $$y'' + 2y' + 2y = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & t \geq \pi \end{cases}$$ $y(0) = 0$, $y'(0) = 1$

$$y'' + 2y' + 2y = u_\pi(t)$$

$$\underbrace{s^2Y - sy(0) - y'(0)}_{\mathcal{L}\{y''\}} + 2\underbrace{(sY - y(0))}_{\mathcal{L}\{y'\}} + 2Y = \frac{e^{-\pi s}}{s}$$

$$(s^2+2s+2)Y = 1 + \frac{e^{-\pi s}}{s}$$

$$Y = \frac{1}{s^2+2s+2} + e^{-\pi s}\boxed{\frac{1}{s(s^2+2s+2)}}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + (Bs+C)s$$

$$1 = (A+B)s^2 + (2A+C)s + 2A$$

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -1$$

$$Y = \frac{1}{s^2+2s+2} + e^{-\pi s}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+2s+2} - \frac{1}{s^2+2s+2}\right)$$

$$Y = \frac{1}{s^2+2s+2} + e^{-\pi s}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+2s+2} - \frac{1}{s^2+2s+2}\right)$$

$$= \frac{1}{(s+1)^2+1} + e^{-\pi s}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right)$$

$$y = e^{-t}\sin t + u_\pi(t)\cdot\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\right)$$

$$Y = \frac{1}{(s+1)^2+1} + e^{-\pi s}\left(\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right)$$

$$y = e^{-t}\sin t + u_\pi(t)\cdot\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - e^{-(t-\pi)}\sin(t-\pi)\right)$$

10. $y(t) - \int_0^t \underbrace{(t-\tau)}_{f(t)=t}\,\underbrace{y(\tau)}_{g(t)=y(t)}\,d\tau = 1$ find $y(2)$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

$$Y - Y\cdot\frac{1}{s^2} = \frac{1}{s}$$

$$Y = \frac{1}{s\left(1-\frac{1}{s^2}\right)} = \frac{s}{s^2-1}$$

$y = \cosh(t)$ $\qquad$ $y(2) = \cosh(2)$

11. another convolution, skip for now

12. $y'' + 2y' + 2y = \frac{1}{2}\delta(t - \frac{\pi}{6}) \qquad y(0)=0, \quad y'(0)=1$

$$s^2Y - \cancel{sy(0)} - y'(0) + 2(sY - \cancel{y(0)}) + 2Y = \frac{1}{2}e^{-\frac{\pi}{6}s}$$

$$(s^2+2s+2)Y = 1 + \frac{1}{2}e^{-\frac{\pi}{6}s}$$

$$Y = \frac{1}{s^2+2s+2} + \frac{1}{2}e^{-\frac{\pi}{6}s}\frac{1}{s^2+2s+2}$$

$s^2+2s+2 = (s+1)^2+1$

$$y = e^{-t}\sin t + \frac{1}{2}u_{\frac{\pi}{6}}(t)\cdot e^{-(t-\frac{\pi}{6})}\sin(t-\tfrac{\pi}{6})$$

13. $F(s) = \dfrac{2s+5}{s^2+2s+10} = \dfrac{2s+5}{(s+1)^2+9} = \dfrac{2(s+1)}{(s+1)^2+9} + \dfrac{3}{(s+1)^2+9}$

$$= 2\cdot\frac{s+1}{(s+1)^2+3^2} + \frac{3}{(s+1)^2+9}$$

$$f(t) = 2e^{-t}\cos 3t + e^{-t}\sin 3t$$