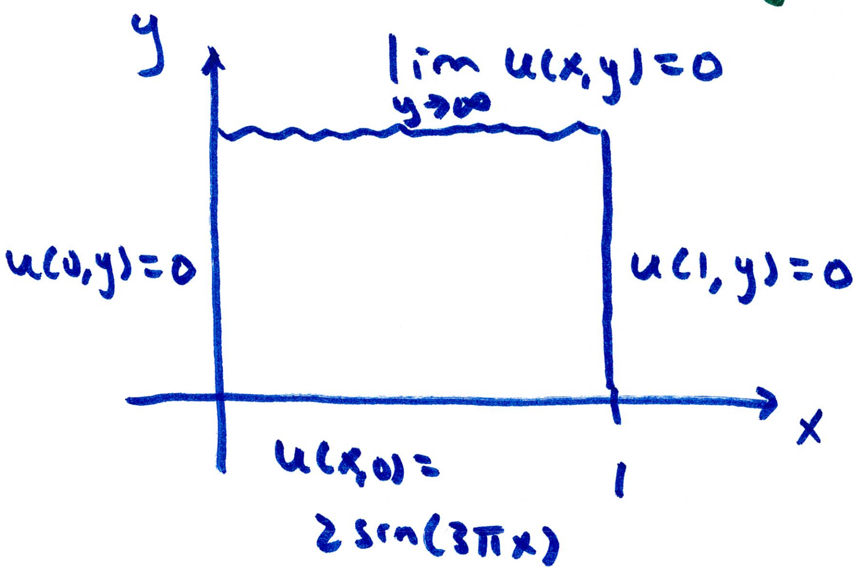
Final exam: Thu. 8/3 8 AM WTHR 172

26. $u_{xx} + u_{yy} = 0 \qquad 0 \le x \le 1 \qquad y \ge 0$

$u(0,y) = 0, \quad u(1,y) = 0$

$u(x,0) = 2\sin(3\pi x) \qquad \lim_{y\to\infty} u(x,y) = 0$



~~Solve for~~

$u = X(x)\,Y(y)$

solve for the one w/ homogeneous BC's first

$u(0,y)=0 \rightarrow X(0)Y(y)=0 \rightarrow X(0)=0$

$u(1,y)=0 \rightarrow X(1)Y(y)=0 \rightarrow X(1)=0$

$u_{xx} + u_{yy} = 0$

$X''Y + XY'' = 0$

$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$ because X must have positive eigenvalues

$X'' + \lambda X = 0 \qquad X(0) = X(1) = 0$

$\lambda_n = \left(\frac{n\pi}{1}\right)^2 = n^2\pi^2$

$X_n = \sin\left(\frac{n\pi x}{1}\right) = \sin(n\pi x) \qquad n = 1, 2, 3, \ldots$

$Y'' - \lambda Y = 0$

$Y'' - n^2\pi^2 Y = 0$

$Y(y) = k_1 e^{n\pi y} + k_2 e^{-n\pi y}$

$\lim_{y\to\infty} u = 0 \qquad \lim_{y\to\infty} X(x) Y(y) = 0$

$X(x) \cdot \left(\lim_{y\to\infty} Y(y) = 0 \right)$

so $k_1 = 0$

$$Y_n = e^{-n\pi y} \qquad u_n = e^{-n\pi y} \sin(n\pi x)$$

$$u(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi y} \sin(n\pi x)$$

$$u(x,0) = 2\sin(3\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$$C_n = \frac{2}{1} \int_0^1 2\sin(3\pi x) \cdot \sin(n\pi x)\, dx$$

better way

$$2\sin(3\pi x) = C_1 \sin(\pi x) + C_2 \sin(2\pi x) + C_3 \sin(3\pi x) + C_4 \sin(4\pi x) + \ldots$$

so $C_3 = 2$, all other C's are zero

$$\boxed{u(x,y) = 2e^{-3\pi y} \sin(3\pi x)}$$

25. $u_{xx} = u_{tt}$ $\quad u(0,t) = u(1,t) = 0$

$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x)$

$u_t(x,0) = 0$

$u = XT$ Solve for the one w/ homogeneous BC's first

$u(0,t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$

$u(1,t) = 0 \rightarrow X(1) = 0$

$$X''T = XT''$$

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda$$

$X'' + \lambda X = 0$ $\quad X(0) = X(1) = 0$

$\lambda_n = \left(\frac{n\pi}{1}\right)^2 = n^2\pi^2$ $\quad X_n = \sin(n\pi x)$

$T'' + \lambda T = 0$ $\quad u_t(x,0) = X(x)T'(0) = 0$

$T'' + n^2\pi^2 T = 0$ $\quad T'(0) = 0$

$$T = k_1 \cos(n\pi t) + k_2 \sin(n\pi t)$$

$$T' = -n\pi k_1 \sin(n\pi t) + n\pi k_2 \cos(n\pi t)$$

$$0 = n\pi k_2 \quad \rightarrow \quad k_2 = 0$$

$$T_n = \cos(n\pi t) \qquad u_n = \cos(n\pi t)\sin(n\pi x)$$

$$u(x,y) = \sum_{n=1}^{\infty} C_n \cos(n\pi t)\sin(n\pi x)$$

$$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) = \sum_{n=1}^{\infty} C_n \sin(n\pi x)$$

$C_5 = 1, \quad C_7 = 2$ all others are zero

$$u(x,y) = \cos(5\pi t)\sin(5\pi x) + 2\cos(7\pi t)\sin(7\pi t)$$

24. $u_{xx} = 2u_t$ $\quad 0 \le x \le 6 \quad t > 0$

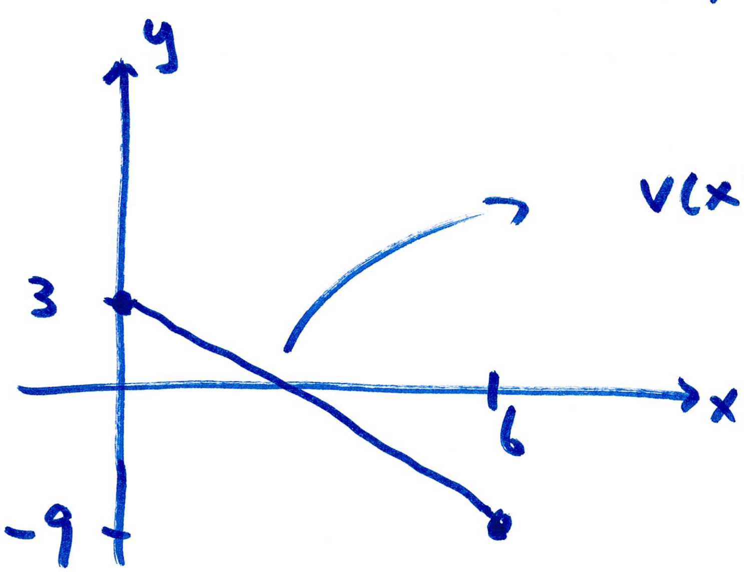
$u(0,t) = 3 \quad u(6,t) = -9$

$u(x,0) = \sin(6\pi x)$

find <u>steady-state</u> solution

if ends not insulated, steady state is line connecting end temps.

if insulated, steady state is average of $u(x,0)$

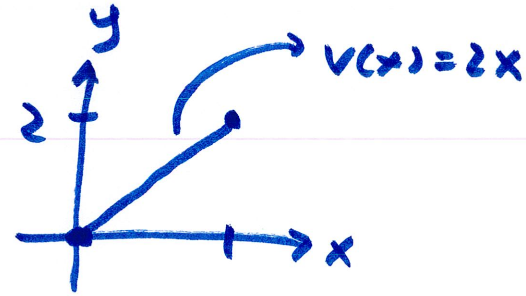


$v(x) = 3 - 2x$

23. $u_{xx} = u_t \qquad 0 \le x \le 1 \qquad t > 0$

$u(0,t) = 0, \qquad u(1,t) = 2$

$u(x,0) = 2x + \sin(\pi x)$



nonhomogeneous BC's

let $w = u - v$

then $w(0,t) = 0, \quad w(1,t) = 0 \qquad$ homogeneous.

still satisfy ~~$u_{xx} = u$~~ heat eq.

$$w_{xx} = u_{xx} - v_{xx} = u_{xx}$$

$$w_t = u_t - v_t = u_t$$

$$w = X T$$

$$X'' T = X T' \qquad X(0) = 0, \quad X(1) = 0$$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0 \qquad X(0) = X(1) = 0$$

$$\lambda_n = n^2\pi^2 \qquad X_n = \sin(n\pi x)$$

$$T' + n^2\pi^2 T = 0$$

$$T_n = e^{-n^2\pi^2 t} \qquad u_n = e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,t) = \underbrace{2x}_{v(x)} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t}\sin(n\pi x)$$

$$u(x,0) = 2x + \sin(\pi x) = 2x + \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

$c_1 = 1$, others $= 0$

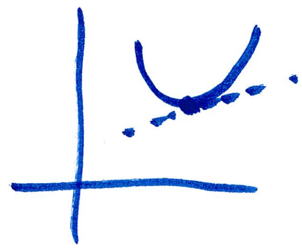
$$u(x,t) = 2x + e^{-\pi^2 t}\sin(\pi x)$$

21.

$y' = 1 - t + 6y \qquad y(1) = 2 \qquad y(2) = ?$

under- or over- estimate ?

if $y'' > 0$



tangent line <u>below</u> graph

Euler always gives

<u>under</u>-estimate

~~indepede~~ <u>independent of h</u>

$y'' = -1 + 6y'$

$= -1 + 6(1 - t + 6y)$

$y'' = 5 - 6t + 36y$

at $t=1$, $y=2$ $\quad y'' > 0$ so tangent line below graph

<u>under</u>-~~st~~estimate

20. $$\vec{x}' = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 9 \\ -1 & -5-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(-5-\lambda)+9=0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda+2)^2 = 0 \qquad \lambda = -2, -2$$

$\underline{\lambda = -2}$

$$\left[\begin{array}{cc|c} 3 & 9 & 0 \\ -1 & -3 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

need to find generalized eigenvector $\vec{u}$: $(A - \lambda I)\vec{u} = \vec{v}$

$$\left[\begin{array}{cc|c} 3 & 9 & 3 \\ -1 & -3 & -1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$u_2 = r \qquad u_1 = 1 - 3r \qquad \vec{u} = \cancel{r\begin{bmatrix} -3 \\ 1 \end{bmatrix}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

same as $\vec{v}$

$$\vec{x}(t) = c_1 e^{-2t}\begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 e^{-2t}\left\{\begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

19. $\vec{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 4t \\ -2e^{t} \end{bmatrix}$

homogeneous part first

$\lambda = 2, \lambda = 3$

$\underline{\lambda=2}$ $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda=3}$ $\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x}_h = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Psi = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ solution: $\vec{x} = \Psi \vec{u}$

$\Psi \vec{u}' = \vec{g}$

$\left[\begin{array}{cc|c} e^{2t} & 0 & 4t \\ 0 & e^{3t} & -2e^{t} \end{array}\right]$ $u_1' = 4te^{-2t}$ ← by parts

$u_2' = -2e^{-2t}$

$u_1 = -2te^{-2t} - e^{-2t} + c_1$

$u_2 = e^{-2t} + c_2$

$\vec{x} = \vec{x}_h + \vec{x}_p$

get this only if

$c_1 = c_2 = 0$