5.1 Review of Power Series

Ch. 5 is on solutions of $P(x)y'' + Q(x)y' + R(x)y = 0$

- you already know how to solve cases where P, Q, R are constants.

e.g. $y'' - y = 0$

characteristic eq:

$r^2 - 1 = 0$

$r = \pm 1$

general solution

$y = c_1 e^{-x} + c_2 e^{x}$

Does NOT work if P, Q, R are NOT constants

want solution in the form of

$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Review of basics of power series

Convergence : $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges ~~iff~~ at a point x if $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ exists, or is defined at x.

Power series converges absolutely if $\sum_{n=0}^{\infty} \left| a_n (x-x_0)^n \right|$ converges.

absolute convergence implies convergence important because the Ratio Test can tell us if series converges ~~abs~~ absolutely.

Ratio Test

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{converges if}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| < 1$$

example

$$\sum_{n=0}^{\infty} \underbrace{\frac{n}{2^n}}_{a_n} \overbrace{x^n}^{(x-x_0)^n} = 0 + \frac{1}{2}x + \frac{2}{2^2}x^2 + \frac{3}{2^3}x^3 + \cdots$$

for what values of x does this converge?

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \cdot x \right| < 1$$

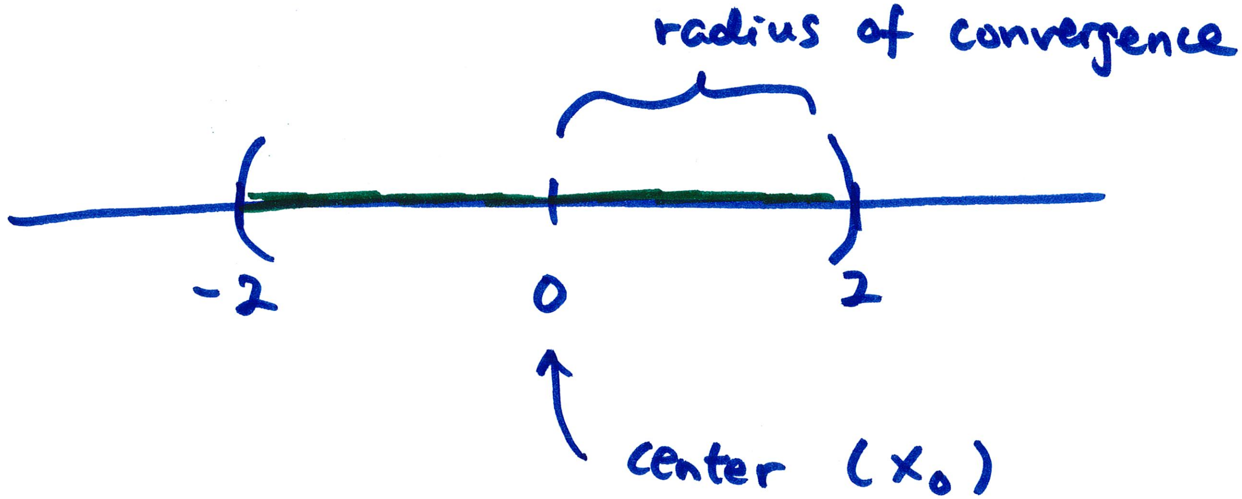
$$\lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{1}{2} \cdot x \right| < 1$$

$$\left| \frac{x}{2} \right| < 1$$

$$-1 < \frac{x}{2} < 1$$

$$\underbrace{-2 < x < 2}_{\text{interval of convergence}}$$

(end points harder to tell if converges, but not important for us)



Taylor Series (near x_0)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

recall $n! = (n)(n-1)(n-2)\cdots\cdots(1)$

$3! = 3 \cdot 2 \cdot 1$

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

$0! = 1$

<u>example</u> Taylor series of e^{-x} near $x_0 = 0$

$f(x) = e^{-x}$ $\qquad$ $f(x_0) = f(0) = 1$

$f'(x) = -e^{-x}$ $\qquad$ $f'(0) = -1$

$f''(x) = e^{-x}$ $\qquad$ $f''(0) = 1$

$f'''(x) = -e^{-x}$ $\qquad$ $f'''(0) = -1$

$f^{(4)}(x) = e^{-x}$ $\qquad$ $f^{(4)}(0) = 1$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = (-1)^n e^{-x}$ $\qquad$ $f^{(n)}(0) = (-1)^n$

$$\text{T.S. of } e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$= 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots$$

Shift of Summation Index

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

n: Summation index

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \underbrace{a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots}$$

could also be written as

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

another way to shift: $\sum_{n=1}^{\infty} n a_n x^{n-1}$

let $m = n-1$, $n = m+1$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$a_1 + 2a_2 x + 3a_3 x^2 + \ldots$	$0 + a_1 + 2a_2 x + 3a_3 x^2 + \ldots$