6.4 Differential Eqs. with Discontinuous Forcing Functions

exam covers up to 6.3 – does NOT include this.

Example

$$y'' + y = \begin{cases} 1 & \pi \le t < 2\pi \\ 0 & 0 \le t < \pi,\ t \ge 2\pi \end{cases}$$

$y(0) = 0, \quad y'(0) = 1$

w/o LT: solve $y'' + y = 0$ $\quad y(0) = 0,\ y'(0) = 1$

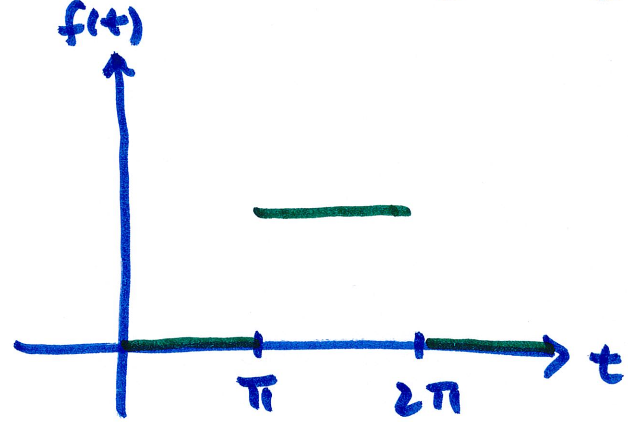
use $y(\pi)$ and $y'(\pi)$ as initial conditions to solve

$y'' + y = 1$

use $y(2\pi)$ and $y'(2\pi)$ as IC's to solve $y'' + y = 0$

$y(0)=0, \quad y'(0)=1$

$$y''+y = \begin{cases} 1 & \pi \le t < 2\pi \\ 0 & 0 \le t < \pi, \ t \ge 2\pi \end{cases} = f(t)$$



$$f(t) = u_\pi(t) - u_{2\pi}(t)$$

$$s^2Y - \cancel{sy(0)} - y'(0) + Y = e^{-\pi s}\cdot\frac{1}{s} - e^{-2\pi s}\cdot\frac{1}{s}$$

$$(s^2+1)Y = 1 + e^{-\pi s}\cdot\frac{1}{s} - e^{-2\pi s}\cdot\frac{1}{s}$$

$$Y = \frac{1}{s^2+1} + e^{-\pi s}\cdot\frac{1}{s(s^2+1)} - e^{-2\pi s}\cdot\frac{1}{s(s^2+1)}$$

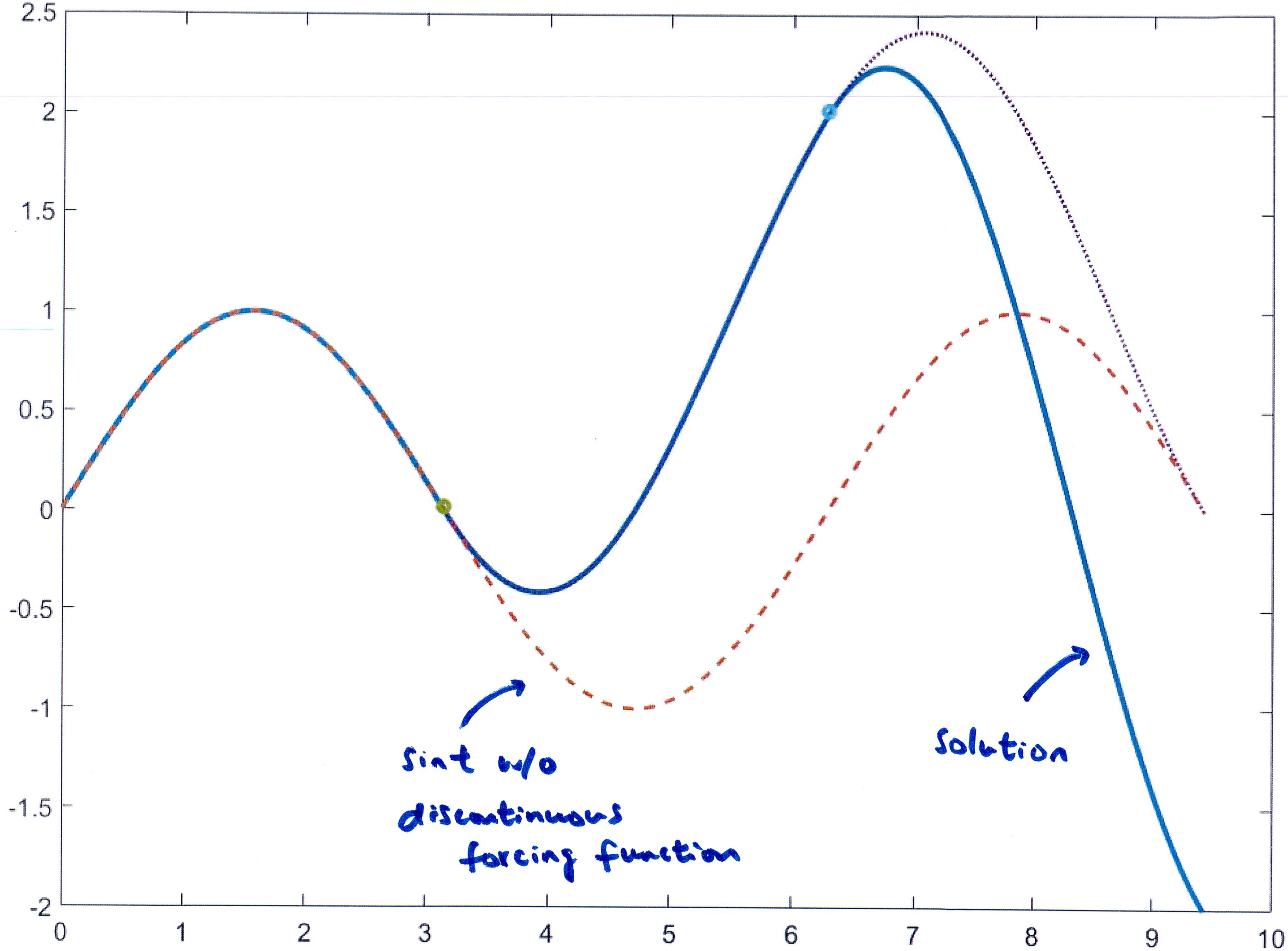
$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$ by partial fraction

$$Y = \frac{1}{s^2+1} + e^{-\pi s}\underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}_{1-\cos(t)} - e^{-2\pi s}\underbrace{\left(\frac{1}{s} - \frac{s}{s^2+1}\right)}$$

$$y(t) = \sin t + u_{\pi}(t)\cdot\left[1-\cos(t-\pi)\right] - u_{2\pi}(t)\cdot\left[1-\cos(t-2\pi)\right]$$

$$= \sin t + u_{\pi}(t)\cdot\left[1+\cos(t)\right] - u_{2\pi}(t)\cdot\left[1-\cos(t)\right]$$

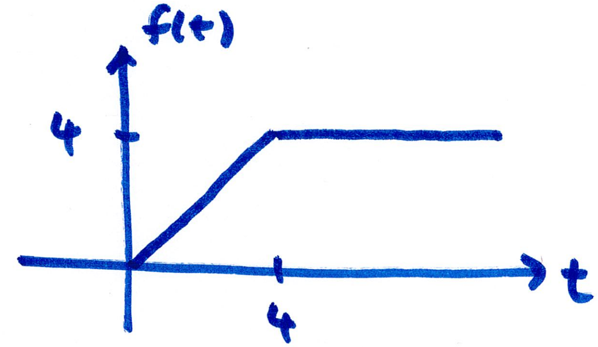
$$= \begin{cases} \sin t & 0 \le t < \pi \\ 1+\sin t+\cos t & \pi \le t < 2\pi \\ \sin t + 2\cos t & \cancel{2\pi}\ \ t \ge 2\pi \end{cases}$$



2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10
sint w/o discontinuous forcing function
Solution

example $y'' - y = f(t)$ $\quad f(t) = \begin{cases} t & 0 \le t < 4 \\ 4 & t \ge 4 \end{cases}$

$y(0) = 0, \quad y'(0) = 1$



$f(t) = t - u_4(t)\cdot(t-4) = t + u_4(t)\,(4-t)$

$s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2} + e^{-4s}\cdot \mathcal{L}\{4-t \text{ shift LEFT by } 4\}$

$s^2Y - 1 - Y = \frac{1}{s^2} + e^{-4s}\mathcal{L}\{4-(t+4)\}$

$= \frac{1}{s^2} + e^{-4s}\left(-\frac{1}{s^2}\right)$

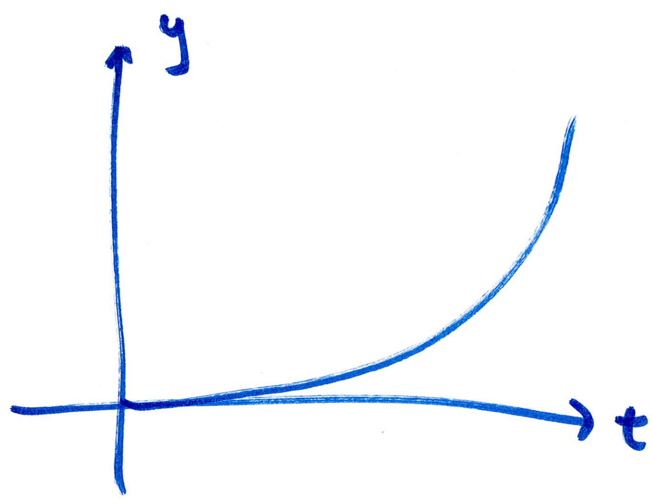
$$(s^2-1)Y = 1 + \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2}\right)$$

$$Y = \underbrace{\frac{1}{s^2-1}} + \underbrace{\frac{1}{s^2(s^2-1)}} - e^{-4s}\left(\frac{1}{s^2(s^2-1)}\right)$$

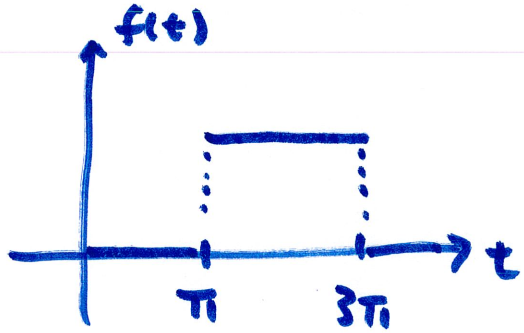
$\sinh(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$

$$\frac{1}{s^2(s^2-1)} = \underbrace{-\frac{1}{s^2} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}}_{\text{inverse LT: } -t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}}$$

$$y(t) = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} - t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - u_4(t)\cdot\left[-(t-4) - \frac{1}{2}e^{-(t-4)} + \frac{1}{2}e^{(t-4)}\right]$$



<u>example</u> $y'' + 4y = u_\pi(t) - u_{3\pi}(t) \qquad y(0) = y'(0) = 0$



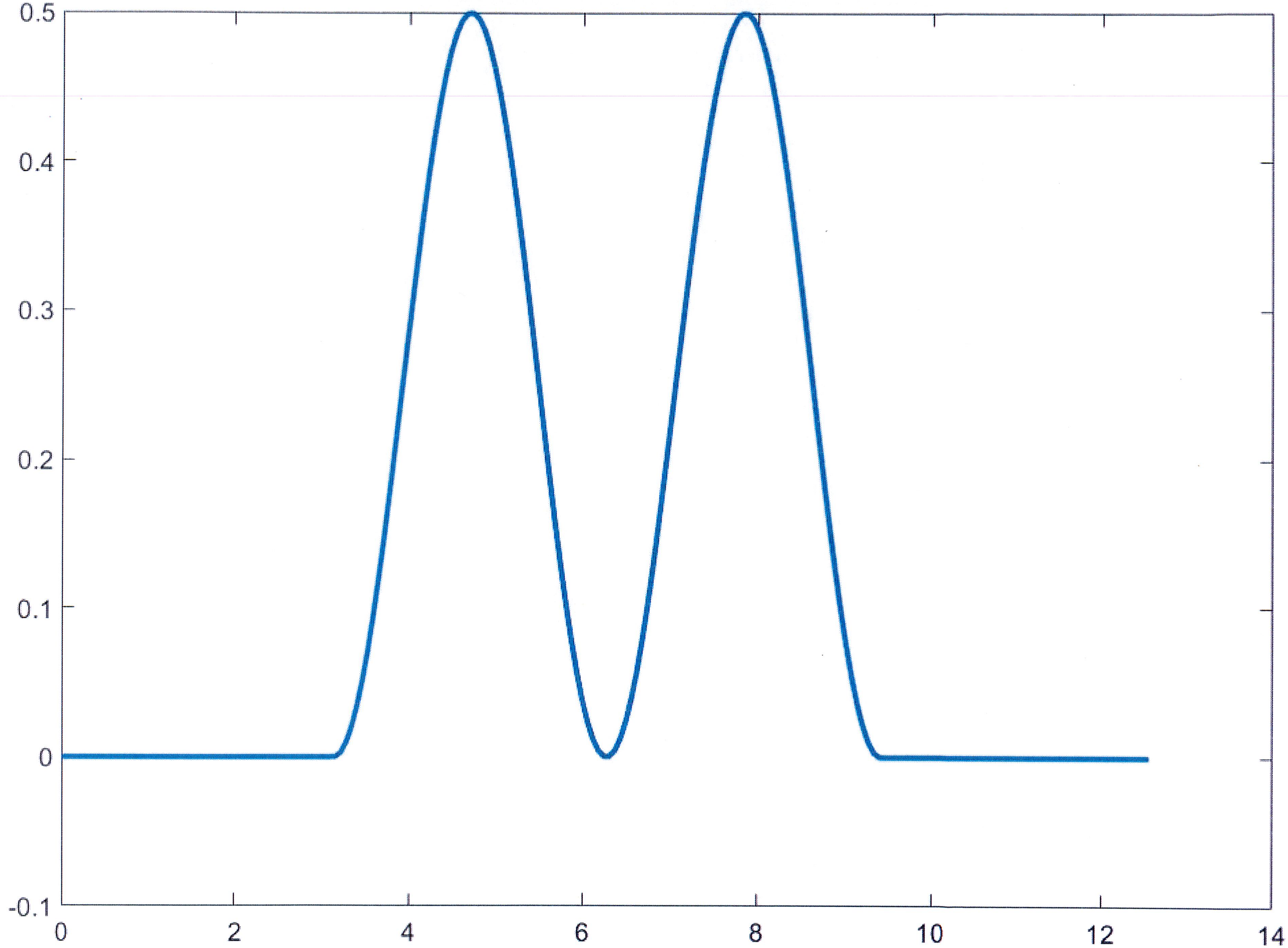
$$(s^2+4)Y = e^{-\pi s}\left(\frac{1}{s}\right) - e^{-3\pi s}\left(\frac{1}{s}\right)$$

$$Y = e^{-\pi s}\underbrace{\left(\frac{1}{s(s^2+4)}\right)} - e^{-3\pi s}\left(\frac{1}{s(s^2+4)}\right)$$

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4}$$

inv. LT: $\frac{1}{4} - \frac{1}{4}\cos 2t$

$$y = u_\pi(t)\cdot\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right] - u_{3\pi}(t)\cdot\left[\frac{1}{4} - \frac{1}{4}\cos 2(t-3\pi)\right]$$


0.5
0.4
0.3
0.2
0.1
0
-0.1
0
2
4
6
8
10
12
14