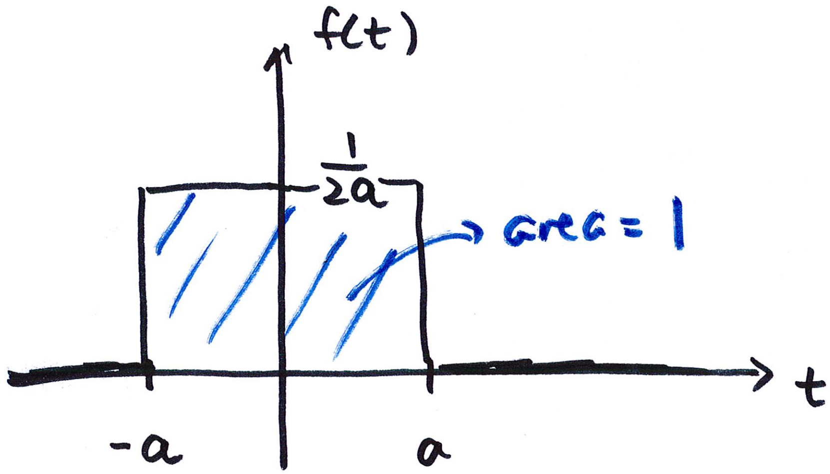
6.5 Impulse Functions

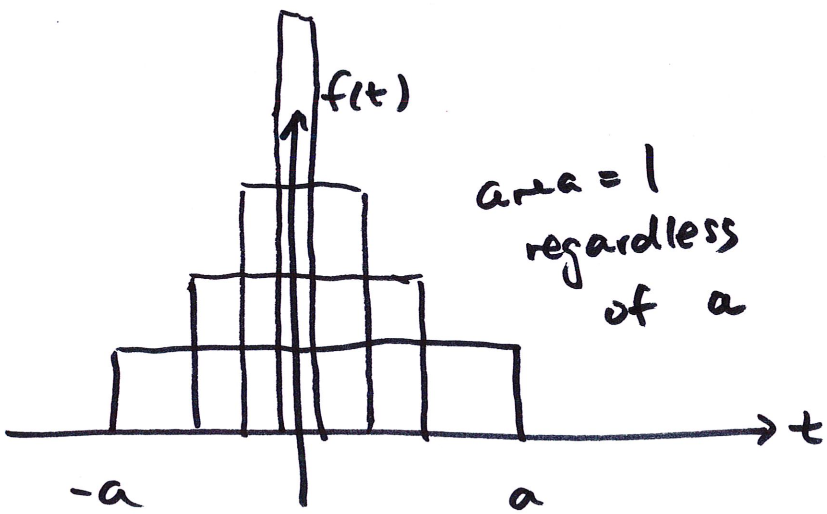
used to model very short-acting force

model from two step functions



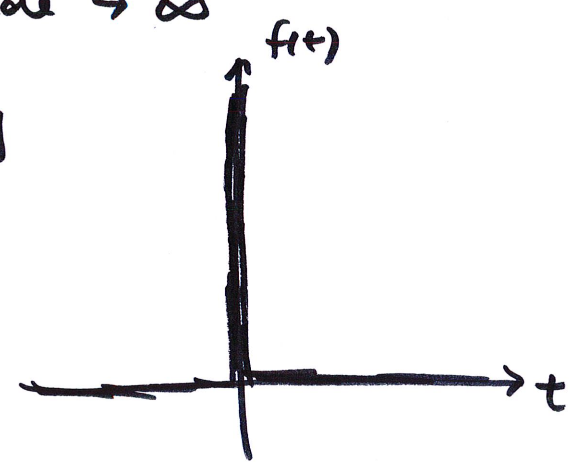
$$f(t) = u_{-a}(t) \cdot \frac{1}{2a} - u_a(t)\frac{1}{2a}$$

turn into a impulse by letting $a \to 0$

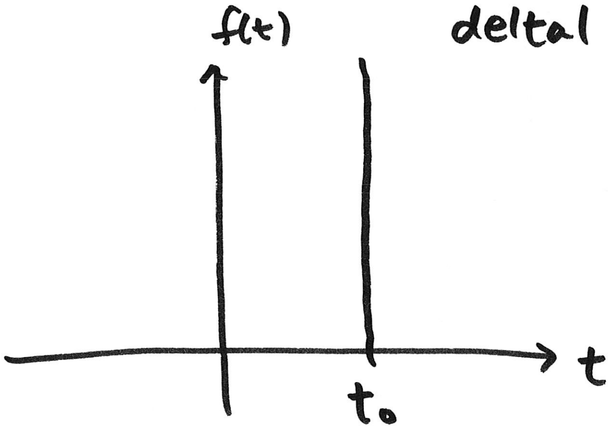


magnitude $\to \infty$

eventually



define impulse function $\quad \delta(t-t_0) = 0 \quad t \neq t_0$



$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

this is also called the Dirac Delta Function

Laplace transform of $\delta(t-t_0)$?

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

u_{t_0+a}

use $\quad \delta(t-t_0) = \lim_{a\to 0}\left(u_{t_0-a}(t)\cdot\frac{1}{2a} - u_{t_0+a}(t)\cdot\frac{1}{2a}\right)$

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0} \mathcal{L}\left\{u_{t_0-a}(t)\cdot\frac{1}{2a} - u_{t_0+a}(t)\cdot\frac{1}{2a}\right\}$$

$$= \lim_{a\to 0}\left[e^{-(t_0-a)s}\frac{1}{2as} - e^{-(t_0+a)s}\frac{1}{2as}\right]$$

$$= \lim_{a\to 0} e^{-t_0 s}\left[\frac{e^{as}}{2as} - \frac{e^{-as}}{2as}\right]$$

$$\frac{e^{as} - e^{-as}}{2} = \sinh(as)$$

$$= e^{-t_0 s} \lim_{a\to 0} \frac{\sinh(as)}{as} = e^{-t_0 s} \lim_{a\to 0} \frac{\cancel{a}\cdot\cosh(as)\cdot\cancel{s}}{\cancel{s}}$$

$$= e^{-t_0 s}$$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}$$

note : $\mathcal{L}\{\delta(t-0)\} = \mathcal{L}\{\delta(t)\} = 1$

<u>example</u> $y'' + y = \delta(t-\pi) \qquad y(0)=0, \quad y'(0)=0$

$$s^2Y - sy(0) - y'(0) + Y = e^{-\pi s}$$

$$(s^2+1)Y = e^{-\pi s}$$

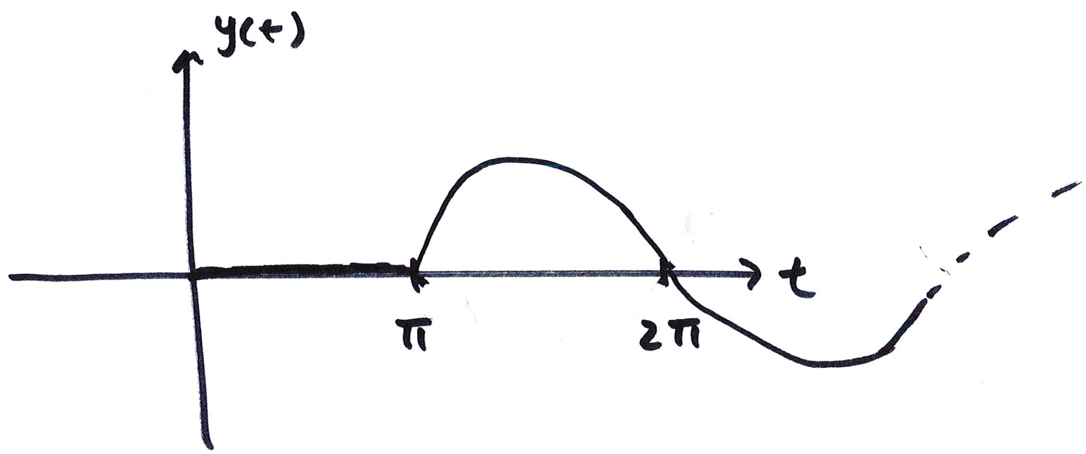
$$Y = e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$y(t) = u_\pi(t) \cdot \sin(t-\pi)$$

no impulse here

<u>unit step</u>

$$y(t) = -u_\pi(t)\sin(t)$$



example $y'' + y = \delta(t-\pi) - \delta(t-2\pi) \quad y(0) = y'(0) = 0$

$$s^2 Y - s y(0) - y'(0) + Y = e^{-\pi s} - e^{-2\pi s}$$

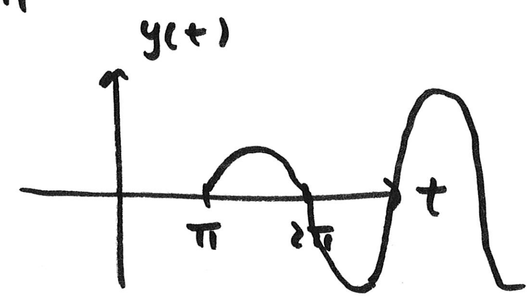
$$(s^2+1) Y = e^{-\pi s} - e^{-2\pi s}$$

$$Y = e^{-\pi s} \cdot \frac{1}{s^2+1} - e^{-2\pi s} \cdot \frac{1}{s^2+1}$$

$$y(t) = u_\pi(t) \cdot \sin(t-\pi) - u_{2\pi}(t) \cdot \sin(t-2\pi)$$

$$= -u_\pi(t) \cdot \sin(t) - u_{2\pi}(t) \cdot \sin(t)$$

$$= \begin{cases} 0 & 0 \le t < \pi \\ -\sin(t) & \pi \le t < 2\pi \\ -2\sin(t) & t \ge 2\pi \end{cases}$$



$$y'' + y = \delta(t-\pi) + \delta(t-2\pi) \qquad y(0) = y'(0) = 0$$

would stop it for $t \geq 2\pi$

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example ~~$y'' + y = \delta(t-\pi)$~~

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t) \qquad y(0)=0, \ y'(0) = \frac{1}{2}$$

$$s^2 Y - s y(0) - y'(0) + 3sY - 3y(0) + 2Y = e^{-5s} + e^{-10s} \cdot \frac{1}{s}$$

$$(s^2 + 3s + 2)Y = \frac{1}{2} + e^{-5s} + e^{-10s} \cdot \frac{1}{s}$$

$$Y = \frac{1}{2} \cdot \frac{1}{(s+1)(s+2)} + e^{-5s} \frac{1}{(s+1)(s+2)} + e^{-10s} \frac{1}{s(s+1)(s+2)}$$

$$= \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) + e^{-5s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

$$+ e^{-10s}\left(\frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \frac{1}{s+2} - \frac{1}{s+1}\right)$$

$$y(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-2t} + u_5(t)\cdot\left[e^{-(t-5)} - e^{-2(t-5)}\right]$$

$$+ u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right]$$

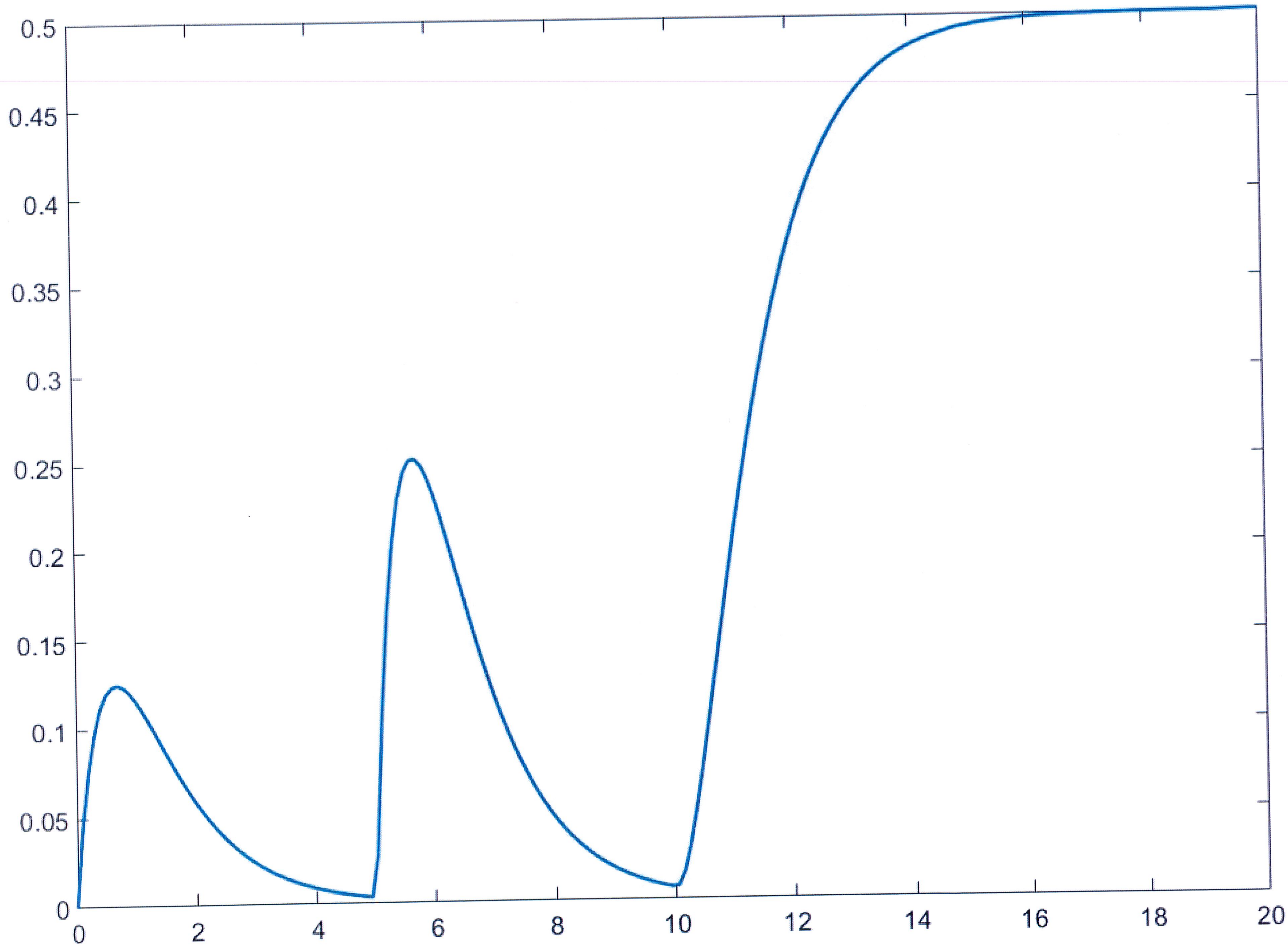
presence of negative exponentials $\Longrightarrow$ as $t \to \infty$, they go to zero

(because of damping)

constant offset of $\frac{1}{2}$ for $t \geq 10$ is due to $u_{10}(t)$ on right side (constant force activated for $t \geq 0$)

(added mass)

as $t \to \infty$, $y \to \frac{1}{2}$


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
12
14
16
18
20