6.6 The Convolution Integral

we know $\mathcal{L}^{-1}\{F(s) \pm G(s)\} = f(t) \pm g(t)$

but $\mathcal{L}^{-1}\{F(s)\,G(s)\} \neq f(t)\,g(t)$ in general

so what is $\mathcal{L}^{-1}\{F(s)\,G(s)\}$?

it turns out (see p. 351-352)

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

τ: dummy variable (for integration)

t is constant for integration purposes

Convolution integral

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= f(t) * g(t)$$

<u>example</u> If $H(s) = \dfrac{1}{(s+1)(s+2)}$ find $h(t)$

"old" way : by partial fraction expansion

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

so $h(t) = e^{-t} - e^{-2t}$

by convolution :

$$H(s) = \frac{1}{(s+1)(s+2)} = \underbrace{\frac{1}{s+1}}_{F(s)} \cdot \underbrace{\frac{1}{s+2}}_{G(s)}$$

$f(t) = e^{-t}$

$g(t) = e^{-2t}$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+2}\right\} = \mathcal{L}^{-1}\{F(s)\cdot G(s)\}$$

$$= \int_0^t e^{-\tau} e^{-2(t-\tau)}\, d\tau$$

$$= \int_0^t e^{-(t-\tau)} e^{-2\tau}\, d\tau$$

$$\int_0^t e^{-\tau} e^{-2t+2\tau}\, d\tau = e^{-2t} \int_0^t e^{-\tau} e^{2\tau}\, d\tau$$

$$= e^{-2t} \int_0^t e^{\tau}\, d\tau = e^{-2t} \left(e^{\tau} \right) \Big|_{\tau=0}^{\tau=t}$$

$$= e^{-2t} \left(e^{t} - 1 \right) = e^{-t} - e^{-2t}$$

<u>example</u> $$\mathcal{L}^{-1}\left\{ \frac{1}{s^4(s^2+1)} \right\} = \mathcal{L}^{-1}\left\{ \underbrace{\frac{1}{s^4}}_{\frac{1}{3!}t^3} \cdot \underbrace{\frac{1}{s^2+1}}_{\sin t} \right\}$$

$$= \int_0^t \frac{1}{6} \tau^3 \sin(t-\tau)\, d\tau \qquad \text{or}$$

$$= \int_0^t \frac{1}{6} (t-\tau)^3 \sin\tau\, d\tau$$

example $y'' - y' - 2y = 4t^2 \quad y(0) = y'(0) = 0$

$$(s^2 - s - 2)Y = 4 \cdot \frac{2}{s^3} = \frac{8}{s^3}$$

$$\underbrace{\boxed{Y}}_{\text{output}} = \boxed{\frac{1}{s^2 - s - 2}} \cdot \underbrace{\boxed{\frac{8}{s^3}}}_{\text{input}}$$

"transfer function" → response of the system due to an impulse at $t=0$

$\mathcal{L}^{-1}\{\text{transfer function}\}$ = impulse response

$$Y = \underbrace{\frac{1}{s^2 - s - 2}}_{\frac{2}{3} e^{\frac{1}{2}t} \sinh\left(\frac{3}{2}t\right)} \cdot \underbrace{\frac{8}{s^3}}_{4t^2}$$

$$\frac{1}{s^2 - s - 2} = \frac{1}{\left(s - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

$$y(t) = \int_0^t \frac{2}{3} e^{\frac{1}{2}\tau} \sinh\left(\frac{3}{2}\tau\right) \cdot 4(t-\tau)^2 \, d\tau$$

$$= \int_0^t \frac{2}{3} e^{\frac{1}{2}(t-\tau)} \sinh\left[\frac{3}{2}(t-\tau)\right] \cdot 4\tau^2 \, d\tau$$

<u>example</u> $\quad y'' + 4y' + 8y = f(t) \qquad y(0) = y'(0) = 0$

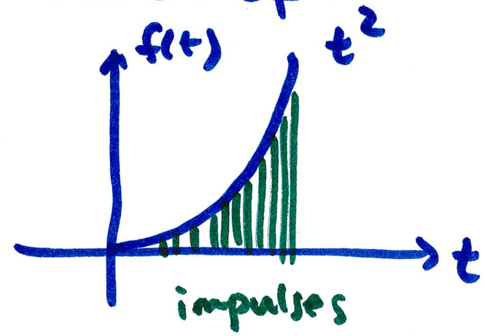
$$(s^2 + 4s + 8) Y = F(s)$$

$$Y(s) = \underbrace{\frac{1}{s^2 + 4s + 8}}_{\substack{\text{complete square} \\ \frac{1}{2} e^{-2t} \sin 2t}} \cdot \underbrace{F(s)}_{f(t)}$$

$$y(t) = \int_0^t \frac{1}{2} e^{-2\tau} \sin 2\tau \cdot f(t-\tau) \, d\tau$$

or $$y(t) = \int_0^t \underbrace{\frac{1}{2} e^{-2(t-\tau)} \sin 2(t-\tau)}_{\text{how system behaves in response to an impulse at } t=\tau} \cdot \underbrace{f(\tau)}_{\text{input/forcing function}} d\tau$$

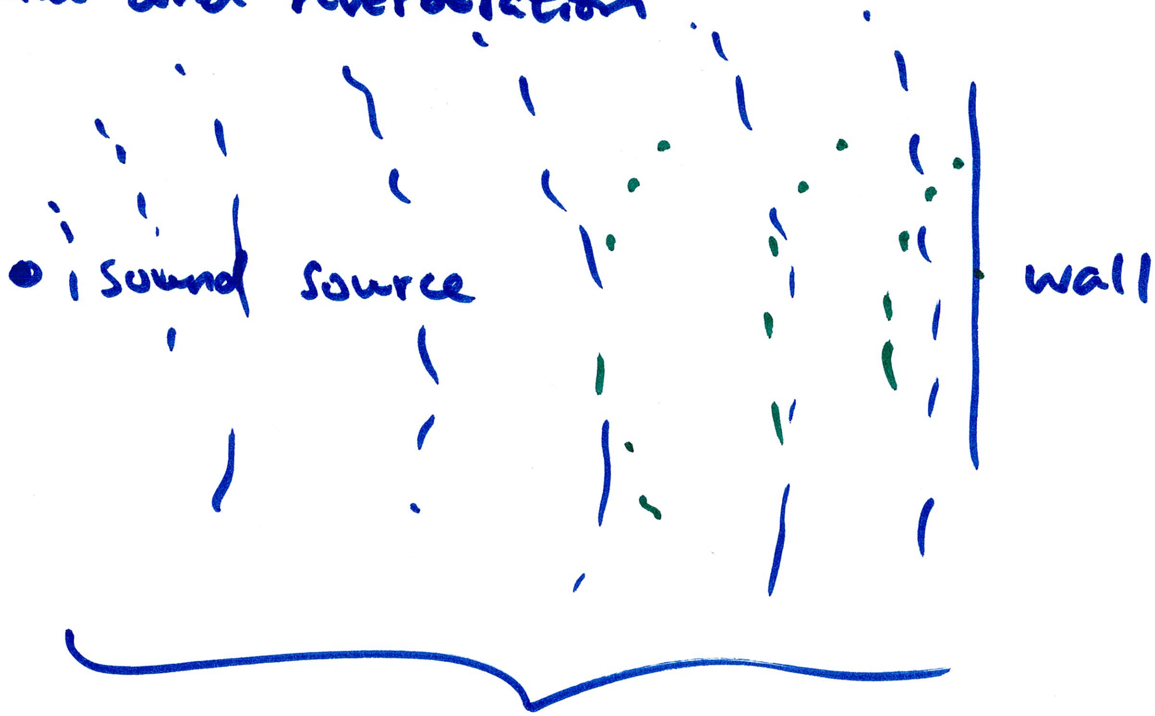
Convolution breaks up $f(t)$ into a bunch of impulses at different times then finds out and combines system response at each impulse time.



$\Rightarrow$ past ~~history~~ histories of system and forcing function are taken into account

Another example application

sound and reverberation



one second
sound travel

Sound is combined/affected by its own echo from one second ago

some kind of convolution of the sound with itself