7.1 + 7.2 Systems of First Order Linear Eqs

Solve multiple DE's simultaneously

$$x_1'(t) = 2x_1(t) + 5x_2(t) \qquad x_1(t) = ?$$

$$x_2'(t) = -x_1(t) + x_2(t) \qquad x_2(t) = ?$$

these equations are <u>coupled</u> because $x_1(t)$ and $x_2(t)$ depend on each other

these are also <u>homogeneous</u> because there are no other functions of t or constants on right side

nonhomogeneous :

$$x_1'(t) = 2x_1(t) + 5x_2(t) + \underline{\cos(t)}$$

$$x_2'(t) = -x_1(t) + x_2(t) - \underline{5}$$

an nth-order DE can be expressed as system of n first-order DE's

<u>example</u> $y'' + \frac{1}{2}y' + 2y = \sin(t)$

2nd order $\rightarrow$ sys of 2 first order eqs.

let $x_1(t) = y$

$x_2(t) = y'$

write DE's for these

$$x_1'(t) = y' = x_2(t)$$

$$x_2'(t) = y'' = -\frac{1}{2}y' - 2y + \sin(t)$$

$$= -\frac{1}{2}x_2(t) - 2x_1(t) + \sin(t)$$

equivalent system:

$$x_1'(t) = x_2(t)$$

$$x_2'(t) = -2x_1(t) - \frac{1}{2}x_2(t) + \sin(t)$$

<u>example</u> $y^{(4)} - y = 0$

4^{th}-order $\rightarrow$ sys. of 4 first-order

$x_1(t) = y$	$x_1' = x_2$
$x_2(t) = y'$	$x_2' = x_3$
$x_3(t) = y''$	$x_3' = x_4$
$x_4(t) = y'''$	$x_4' = x_1$

Reverse is a little more complicated

<u>example</u> $x_1' = -2x_1 + x_2$ — ①

$x_2' = x_1 - 2x_2$ — ②

turn into a 2nd-order eq

from ①, solve for x_2 (appears only once)

$x_2 = x_1' + 2x_1$ sub into ②

$x_1'' + 2x_1' = x_1 - 2(x_1' + 2x_1)$ no x_2 here

$x_1'' + 2x_1' = x_1 - 2x_1' - 4x_1$

$x_1'' + 4x_1' + 3x_1 = 0$ → equivalent 2nd-order eq. in x_1

(can do the same for x_2)

solve $x_1'' + 4x_1' + 3x_1 = 0$

$(r+3)(r+1) = 0 \qquad r=-1,\ r=-3$

$x_1(t) = c_1 e^{-t} + c_2 e^{-3t}$

$x_2 = x_1' + 2x_1$ (from earlier step)

$= -c_1 e^{-t} - 3c_2 e^{-3t} + 2c_1 e^{-t} + 2c_2 e^{-3t}$

$x_2 = c_1 e^{-t} - c_2 e^{-3t}$

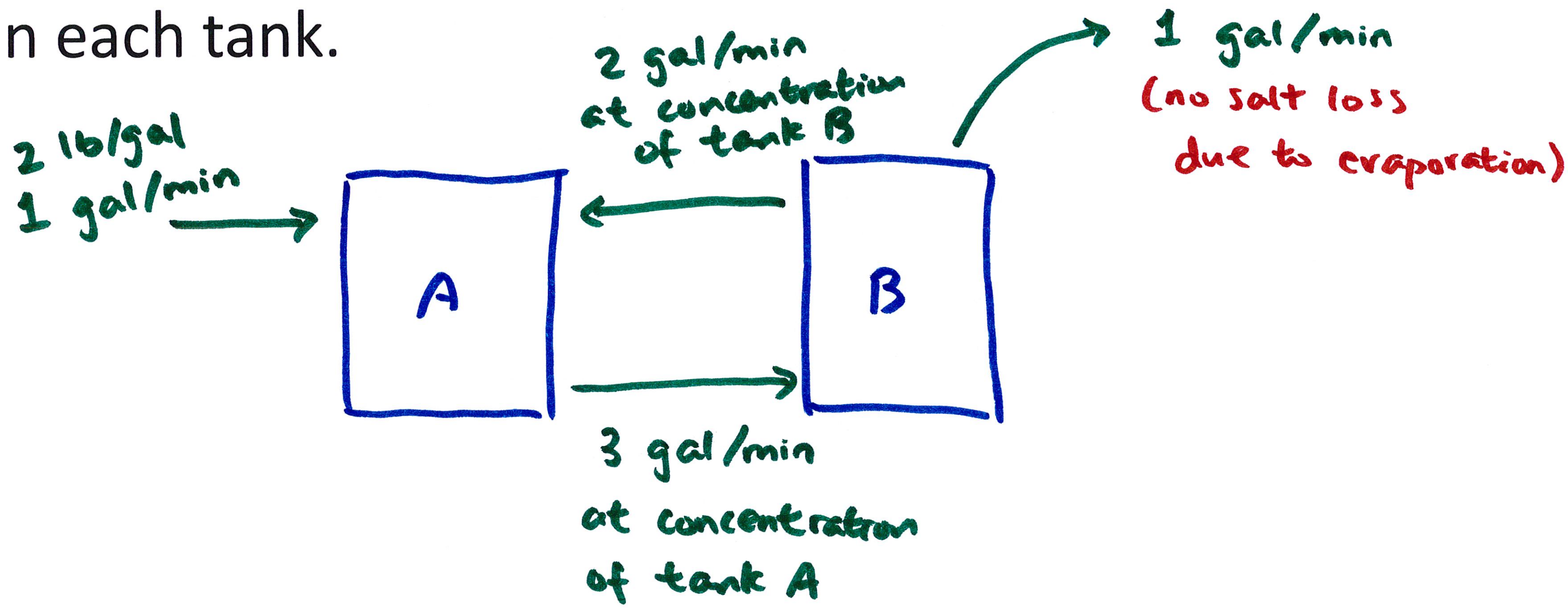
this process can be used (in principle) to solve any system.

but can be cumbersome

usually use techniques based on linear algebra.

another example before linear algebra review.

- There are two 100-gallon tanks both initially full of pure water. Water containing salt at a concentration of 2 lb/gal flows into tank A from an outside source at 1 gal/min. The well-mixed water flows from tank A to tank B at 3 gal/min. Water evaporates from tank B at 1 gal/min and is also piped back to tank A at 2 gal/min. Write a system describing the rate of change of the amount of salt in each tank.



$Q_1(t)$: salt (in lb) in A

$Q_2(t)$: " " " B

$$Q_1' = (\text{rate in}) - (\text{rate out})$$

$$= (2\ \text{lb/gal})(1\ \text{gal/min}) + (2\ \text{gal/min})\left(\frac{Q_2}{100}\right) - (3\ \text{gal/min})\left(\frac{Q_1}{100}\right)$$

$$Q_2' = (3\ \text{gal/min})\left(\frac{Q_1}{100}\right) - (2\ \text{gal/min})\left(\frac{Q_2}{100}\right)$$

$$Q_1' = 2 - \frac{3}{100}Q_1 + \frac{1}{50}Q_2$$

$$Q_2' = \frac{3}{100}Q_1 - \frac{1}{50}Q_2$$

initial conditions: $Q_1(0) = 0$

$Q_2(0) = 0$

Review of Matrices

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \qquad \text{book notation: } \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$$

$$B = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

can add/subtract matrices of the <u>same size</u>

$A \pm B$ are meaningful but $B \pm C$ are not

$$A + B = \begin{bmatrix} 4 & 3 \\ 4 & 1 \end{bmatrix}$$

multiplication: inner dimensions must match

$$\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 27 & -9 \\ 12 & -4 \end{bmatrix}$$

2×2 $\qquad$ 2×2

R $\quad$ C

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$ is meaningless

2×1 $\qquad$ 2×2

not the same

but $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is ok.

inverse of matrices

$$A^{-1}A = I \quad \text{(identity)}$$
$$AA^{-1} = I$$

but recall $AB \neq BA$ in general for arbitrary A, B

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$$

$$A = \left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right]$$

do row ops to make left $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, right is A^{-1}

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 11 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 3/11 & -4/11 \\ 2/11 & 1/11 \end{bmatrix}$