7.3 Linear Independence, Eigenvalues and Eigenvectors

solving linear eqs.

$$x_1 + 2x_2 + 3x_3 = 9$$
$$2x_1 - x_2 + x_3 = 8$$
$$3x_1 \qquad - x_3 = 9$$

$x_1 = ?$
$x_2 = ?$
$x_3 = ?$

in matrix form:
$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix}}_{\vec{b}}$$

$$A\vec{x} = \vec{b}$$

$\vec{b} \neq \vec{0}$ nonhomogeneous

(homogeneous always has the trivial solution $\vec{x} = \vec{0}$)

augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 9 \end{array}\right]$

Solve by Gaussian elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -18 \end{array}\right] \quad \begin{array}{l} \leftarrow -2R_1 + R_2 \\ \leftarrow -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -18 \end{array}\right] \leftarrow -\frac{1}{5}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -6 \end{array}\right] \leftarrow 6R_2 + R_3$$

solve by back substitution

R_3 : $-4x_3 = -6 \rightarrow x_3 = +3/2$

R_2 : $x_2 + x_3 = 2 \rightarrow x_2 = 2 - 3/2 = \frac{1}{2}$

R_1 : $x_1 + 2x_2 + 3x_3 = 9 \rightarrow x_1 = 9 - 2x_2 - 3x_3$

$= 9 - 1 - 9/2 = \frac{7}{2}$

solution: $\vec{x} = \begin{bmatrix} 7/2 \\ 1/2 \\ 3/2 \end{bmatrix}$ unique (only one solution vector)

nonhomogeneous sys can have more than one (or none) solutions.

<u>example</u>

$$x_1 + 2x_2 - x_3 = -2$$
$$-2x_1 - 4x_2 + x_3 = 4$$
$$2x_1 + 4x_2 - 2x_3 = -4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 1 & 4 \\ 2 & 4 & -2 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & -2 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{row of zeros}$$

(infinitely many solutions)
one variable is "free"

leading 1 in column 1
and column 3

no leading 1 in column 2

$\Rightarrow$ x_2 to be free variable

$x_3 = 0$ (from row 2)

$x_2 = r$

$$x_1 + 2x_2 - x_3 = -2$$

$$x_1 = -2 - 2x_2 = -2 - 2r$$

solution: $x_1 = -2 - 2r$

$x_2 = r$ $\qquad$ r = any real number

$x_3 = 0$

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what if we have

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$\rightarrow$ not possible for any ~~real~~ x_1, x_2, x_3

so <u>no solutions</u>

linear independence

$\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_n$ are <u>linearly independent</u>

if $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \dots + c_n\vec{x}_n = \vec{0}$

implies $c_1 = c_2 = c_3 = \dots = c_n = 0$

<u>example</u> $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

linearly independent if $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has the only solution $c_1 = c_2 = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$c_2 = 0$, $c_1 = 0$

so are linearly independent

<u>example</u> $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

solve by finding solutions to

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

$c_2 = 3$

$c_1 = -1$

$c_1 \neq 0, \; c_2 \neq 0$

NOT independent

<u>example</u> $\vec{x}_1 = \begin{bmatrix} 2\sin t \\ \sin t \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} \sin t \\ 2\sin t \end{bmatrix}$

solve $c_1 \vec{x}_1 + c_2 \vec{x}_2 = \vec{0}$ for c_1, c_2

$$\left[\begin{array}{cc|c} 2\sin t & \sin t & 0 \\ \sin t & 2\sin t & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \sin t & 2\sin t & 0 \\ 2\sin t & \sin t & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} \sin t & 2\sin t & 0 \\ 0 & -3\sin t & 0 \end{array}\right]$$

$-3\sin t \cdot c_2 = 0 \rightarrow c_2 = 0$ (for all t)

$\sin t \cdot c_1 = 0 \rightarrow c_1 = 0$

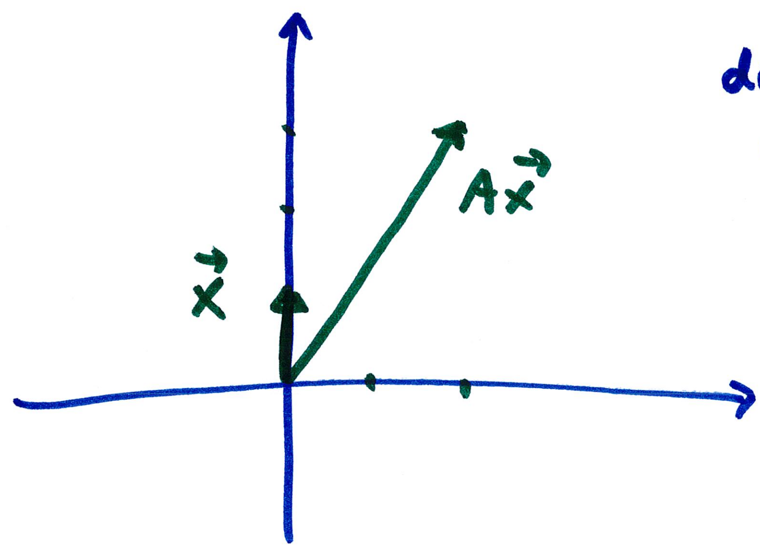
so independent

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$A\vec{x}$ is a linear transformation of $\vec{x}$

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



direction has been changed

some vectors preserve their directions $\Rightarrow$ <u>eigenvectors</u>

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

direction is preserved

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A

Sometimes eigenvectors change their magnitudes by ~~the~~ a factor equal to <u>eigenvalue</u>

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\hookrightarrow$ eigenvalue associated w/ eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

How to find them?

if $\vec{x}$ is an eigenvector and λ is the associated eigenvalue, then $\underbrace{A\vec{x} = \lambda\vec{x}}_{\text{solve for } x \text{ and } \lambda}$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\boxed{(A - \lambda I)\vec{x} = \vec{0}} \leftarrow \text{solve}$$

homogeneous sys, trivial solution $\vec{x} = \vec{0}$ always exists BUT we don't want it.

nontrivial solution exists if there is a zero row in $A - \lambda I$

$$\Rightarrow \boxed{\det(A - \lambda I) = 0}$$

example $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(3-\lambda) = 0$$

$$\lambda = 1, \quad \lambda = 3$$

solve $(A - \lambda I)\vec{x} = \vec{0}$ for each λ

$\lambda = 1$ $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_2 = 0 \\ x_1 = r \end{array}$$

eigenvector: $\vec{x} = \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

choose r to be something convenient

$\underline{\lambda=3}$ $(A-\lambda I)\vec{x}=\vec{0}$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = r$

$x_1 = x_2 = r$

$\vec{x} = \begin{bmatrix} r \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ has $\lambda = 1,\ \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3,\ \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

what about $A = \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 4 \\ 0 & 6-\lambda \end{vmatrix} = 0$ $\quad (2-\lambda)(6-\lambda)=0 \to \lambda=2,\ \lambda=6$

$\underline{\lambda=2}$ $\left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & 4 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda=6}$ $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so $cA = B$ and A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$

eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

the B has eigenvalues $c\lambda_1, c\lambda_2, \ldots, c\lambda_n$

eigenvectors same as those of A