7.5 Homogeneous Systems with Constant Coefficients

Solve $\vec{x}' = A\vec{x}$ $\quad$ A : constant $n \times n$ matrix

$y'' + 5y' + 6y = 0$ is equivalent to

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{x}$$

$(\vec{x}_1 = y, \ \vec{x}_2 = y')$

should have the <u>same solution</u>

characteristic eq: $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, \ r = -3$

Solution $y = c_1 e^{-2t} + c_2 e^{-3t} \rightarrow x_1(t)$ in $\vec{x}' = A\vec{x}$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \rightarrow x_2(t)$ in $\vec{x}' = A\vec{x}$

rewrite: $\begin{bmatrix} y \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{\text{solution of } \vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{x}}$

eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-5-\lambda) + 6 = 0$$

$$\lambda^2 + 5\lambda + 6 = 0$$

characteristic eq

$$\lambda = -2, \quad \lambda = -3$$

$\underline{\lambda = -2}$ $\quad (A - \lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_2 = r$

$v_1 = -\frac{1}{2} r$

$$\vec{v} = r \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (r = -2)$$

Similarly, $\lambda = -3$ has $\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

In general, $\vec{x}\,' = A\vec{x}$ where A is a constant $n \times n$ matrix

solution is $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \cdots + c_n e^{\lambda_n t} \vec{v}_n$

λ_i : eigenvalues

$\vec{v}_n$: corresponding ~~eigenvalues~~ eigenvectors

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \vec{x}$$

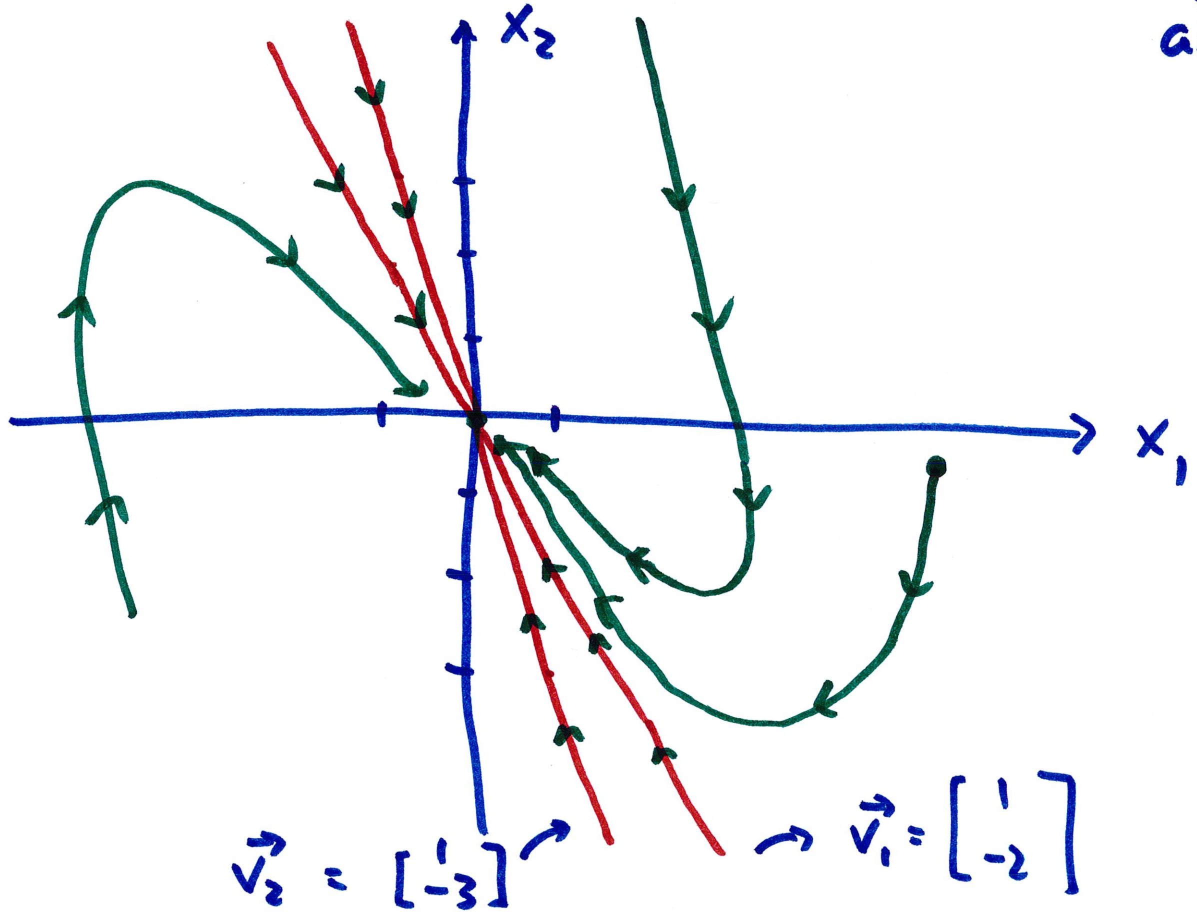
has solution $\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

note: $t \to \infty$ $\vec{x} = \vec{0}$

origin is a asymptotically stable node

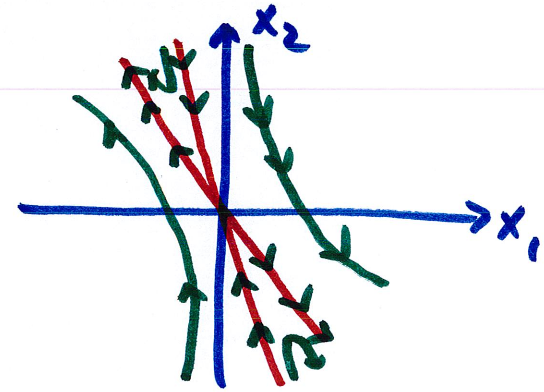
phase portrait (plot of x_1 vs x_2)



if eigenvalue is positive, solution will move away from origin

$\lambda_1 > 0 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 < 0 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$



origin is a <u>saddle point</u>

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \vec{x}$$

find eigenvalue / eigenvector pairs

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 4 & -2-\lambda \end{vmatrix} - \begin{vmatrix} -2 & 3-\lambda \\ 4 & -1 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)(-2-\lambda)+2\right] + \left[(-2)(-2-\lambda) - 8\right] - \left[(-2)(-1) - (4)(3-\lambda)\right] = 0$$

$$(3-\lambda)(\lambda^2-\lambda-4) + (2\lambda-4) - (4\lambda-10) = 0$$

$$3\lambda^2 - 3\lambda - 12 - \lambda^3 + \lambda^2 + 4\lambda + 2\lambda - 4 - 4\lambda + 10 = 0$$

$$-\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + \boxed{6} = 0$$

factors: $1, 6$ or $-1, -6$

$2, 3$ or $-2, -3$

at least one is a root

try $\lambda = 1$

$$1 - 4 + 1 + 6 \neq 0$$

try $\lambda = -1$

$$-1 - 4 - 1 + 6 = 0$$

$\lambda = -1$ is a solution

$$(\lambda+1)(\quad ? \quad)(\quad ? \quad) = 0$$

rearrange $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$ to factor $\lambda + 1$ by grouping

$$\lambda^3 + \lambda^2 - 5\lambda^2 - 5\lambda + 6\lambda + 6 = 0$$

$$\lambda^2(\lambda+1) - 5\lambda(\lambda+1) + 6(\lambda+1) = 0$$

$$(\lambda+1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda+1)(\lambda-2)(\lambda-3) = 0$$ e-values: $\lambda = -1, 2, 3$

$\underline{\lambda = -1}$ $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 4 & -1 & -1 & 0 \\ -2 & 4 & 2 & 0 \\ 4 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$V_3 = r$

$V_2 = -\frac{3}{7}r$

$V_1 = 2V_2 + V_3 = \frac{1}{7}r$

choose $r = 7$

$$\vec{V} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

Similarly, $\lambda = 2$ $\quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 3$ $\quad \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

solution: $\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\vec{x}' = A\vec{x}$ has solutions of the form $\vec{v} e^{rt}$

just like $y' = ay$ has solution of the form e^{rt}

$t\vec{x}' = A\vec{x}$ is analogous to Euler equation

$$t^2 y'' + \alpha t y' + \beta y = 0$$

solutions $|t|^r$

has solutions of the form $\vec{x} = \vec{v} t^r$

$$t\vec{x}' = A\vec{x}$$

$$\vec{x} = \vec{v}t^r \rightarrow \vec{x}' = \vec{v}rt^{r-1}$$

sub into $t\vec{x}' = A\vec{x}$

$$t\vec{v}rt^{r-1} = A\vec{v}t^r$$

$$r\vec{v}t^r = A\vec{v}t^r \qquad t^r \neq 0 \text{ in general}$$

divide by t^r

$$r\vec{v} = A\vec{v}$$

$$\left.\begin{aligned} (A - rI)\vec{v} &= \vec{0} \\ \det(A - rI) &= 0 \end{aligned}\right\}$$

r : eigenvalues

$\vec{v}$: corresponding eigenvectors

solution: (2×2) $\vec{x}(t) = c_1\vec{v}_1 t^{r_1} + c_2\vec{v}_2 t^{r_2}$