# 7.6 Complex Eigenvalues

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$

eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 + 1 = 0$

$1-\lambda = \pm i$

$\lambda = 1 \pm i$

find eigenvectors:

$\underline{\lambda = 1+i}$ solve $(A-\lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_2 = r$

$v_1 = \frac{1}{i} r$

$\vec{v} = r \begin{bmatrix} \frac{1}{i} \\ 1 \end{bmatrix}$ choose $r = i$

$\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\underline{\lambda = 1 - i}$ Solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_2 = r \qquad v_1 = -\frac{1}{i} r \qquad$ choose $r = -i$

$$\vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Euler's ~~Identi-~~ Formula

$$e^{it} = \cos t + i \sin t$$

solution: $\vec{X}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$

$$\vec{x}_1(t) = e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^t e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t(\cos t + i \sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix} = e^t\left\{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}\right\}$$

$$\vec{x}_2(t) = e^{(1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^t e^{i(-t)}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^t(\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^t\begin{bmatrix} \cos t - i\sin t \\ -\sin t - i\cos t \end{bmatrix} = e^t\left\{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}\right\}$$

fundamental solutions are conjugate pairs

general solution: $\underbrace{\vec{x}(t)}_{\text{real}} = D_1\vec{x}_1 + D_2\vec{x}_2$ $D_1, D_2$ are constants (complex)

$$\vec{x}(t) = D_1 e^t\left\{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}\right\}$$

$$+ D_2 e^t\left\{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} - i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}\right\}$$

$$\vec{x}(t) = \underbrace{(D_1+D_2)}_{c_1}\underbrace{e^t\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}}_{\substack{\text{real part}\\ \text{of either}\\ \text{fundamental}\\ \text{solution}}} + \underbrace{i(D_1-D_2)}_{c_2}\underbrace{\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}e^t}_{\substack{\text{imaginary}\\ \text{part of}\\ \text{either solution}}}$$

in practices, find either $\vec{x}_1$ or $\vec{x}_2$

general solution: $\vec{x}(t) = c_1$ (real part of either)

$+\, c_2$ (imag part of either)

example

$$\vec{x}\,' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

$$\lambda = -1 \pm 2i$$

$\lambda = -1+2i$

$$\left[\begin{array}{cc|c} -2i & -4 & 0 \\ 1 & -2i & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2ir \\ r \end{bmatrix} = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \qquad (r=1)$$

$$\vec{x}_1(t) = e^{(-1+2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$= e^{-t}(\cos 2t + i \sin 2t) \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$= e^{-t} \left\{ \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + i \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} \right\}$$

gen. solution:

$$\vec{X}(t) = c_1 e^{-t} \begin{bmatrix} -2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix}$$

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back to $\vec{v} = \begin{bmatrix} 2ir \\ r \end{bmatrix}$ choice of $r$ will affect appearance of solution

same

choose $r = -i$ $\quad \vec{v} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$

$$\vec{X}_1(t) = e^{(-1+2i)t} \begin{bmatrix} 2 \\ -i \end{bmatrix} = e^{-t}(\cos 2t + i\sin 2t) \begin{bmatrix} 2 \\ -i \end{bmatrix}$$

$$= e^{-t} \left\{ \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} + i \begin{bmatrix} 2\sin 2t \\ -\cos 2t \end{bmatrix} \right\}$$

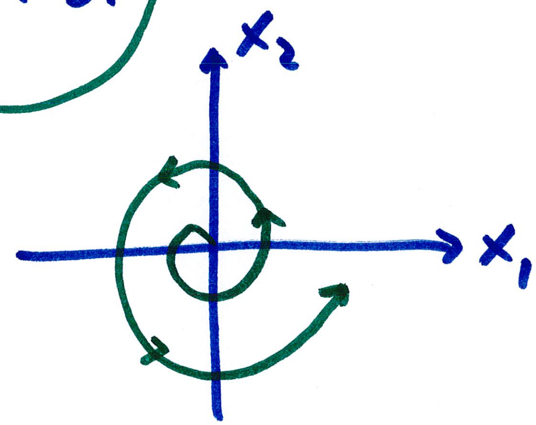
gen. solution:

$$\vec{X}(t) = c_1 e^{-t} \begin{bmatrix} 2\cos 2t \\ \sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\sin 2t \\ -\cos 2t \end{bmatrix}$$

# Phase Portraits and Eigenvalues

if $\lambda = a + bi$ $\qquad$ $e^{at} \to \infty$ as $t \to \infty$

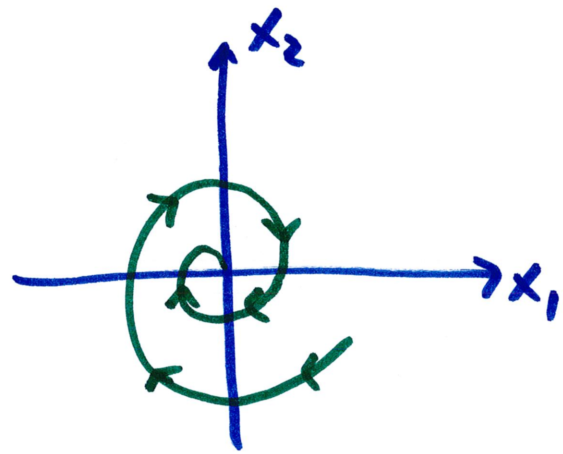
$a > 0$




unstable spiral

($\vec{0}$ is an unstable equilibrium)

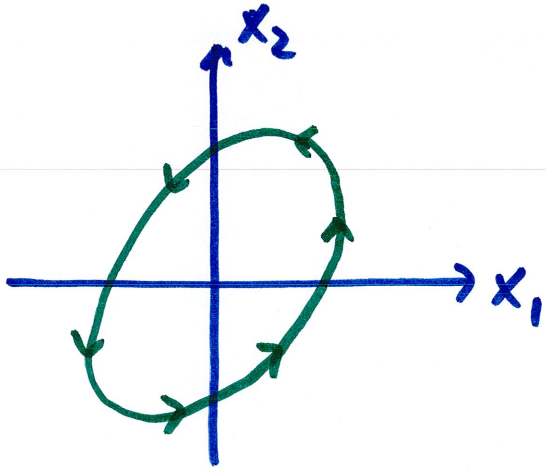
$a < 0$




stable spiral

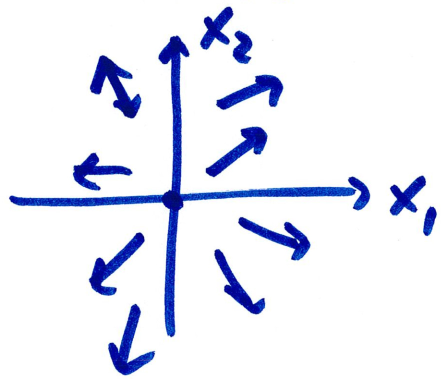
($\vec{0}$ is a asymptotically stable eq.)

$a = 0$



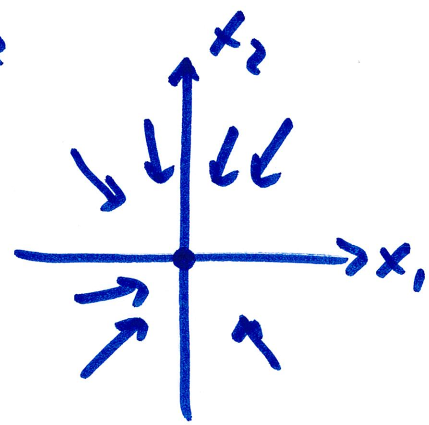

if $\lambda$ are real and distinct

both positive




Origin is unstable eq.

both negative




~~bo~~ opposite signs

origin is saddle pt