HW from 7.6

20. $\vec{x}' = \begin{bmatrix} 4 & \alpha \\ 8 & -6 \end{bmatrix} \vec{x}$

$$\lambda^2 + 2\lambda - 8\alpha - 24 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(-8\alpha - 24)}}{2} = \frac{-2 \pm \sqrt{32\alpha + 100}}{2}$$

λ complex $\rightarrow$ spirals

λ real same sign $\rightarrow$ sink / source at origin

λ real opposite signs $\rightarrow$ saddle point at origin

λ real same sign $\Rightarrow$ $\sqrt{32\alpha + 100} < 2$

$$32\alpha + 100 < 4$$

$$32\alpha < -96$$

$$\alpha < -3$$

7.7 Fundamental Matrices

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$$

has solution $\vec{x}(t) = c_1 \underbrace{e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{x}_2(t)}$

Fundamental solutions
(linearly independent)

can rewrite $\vec{x}(t)$ as

$$\vec{x}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\vec{x}_1(t)$ $\vec{x}_2(t)$

the matrix w/ fundamental solutions as columns

is call a <u>Fundamental Matrix</u> $\Psi(t) = \left[\vec{x}_1(t)\ \vec{x}_2(t) \dots\right]$

so $\vec{x}(t) = \Psi(t)\vec{c}$ is the general solution

usually we know $\vec{x}(t_0)$ and not $\vec{c}$ directly

rewrite $\vec{x}(t)$ to get $\vec{x}(t_0)$ in and remove $\vec{c}$

$\vec{x}(t) = \Psi(t)\vec{c}$

at $t = t_0$, $\vec{x}(t_0) = \Psi(t_0)\vec{c}$

solve for c:

$$\underbrace{\Psi^{-1}(t_0)}\,\vec{x}(t_0) = \vec{c}$$

exists because its columns are fundamental solutions (linearly independent) rewrite : $\vec{x}(t) = \underbrace{\Psi(t)\,\Psi^{-1}(t_0)}\,\vec{x}(t_0)$

~~calls~~ define $\Phi(t) = \Psi(t)\,\Psi^{-1}(t_0)$

is also called a

Fundamental Matrix

(State Transition Matrix)

$$\underbrace{\vec{x}(t)}_{\text{"state" at } t} = \Phi(t)\,\underbrace{\vec{x}(t_0)}_{\text{state at } t=t_0}$$

$$\vec{x}(t_0) = \Phi^{-1}(t)\,\vec{x}(t)$$

also note $\boxed{\Phi(t_0) = I}$

NOT true for

$\Psi(t_0)$ in general

back to $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x}$

$$\vec{x}(t) = \underbrace{\begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}}_{\Psi(t)} \vec{c}$$

find $\Phi(t)$

let $t_0 = 0$ $\qquad \Phi(t) = \Psi(t)\,\Psi^{-1}(t_0)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

find $\Psi^{-1}(t)$

$$\Psi^{-1}(t) = \frac{1}{4e^{t} - e^{t}} \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{2t} & 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3}e^{-2t} & -\frac{1}{3}e^{-2t} \\ -\frac{1}{3}e^{t} & \frac{2}{3}e^{t} \end{bmatrix}$$

$$\Psi^{-1}(t_0) = \Psi^{-1}(0) = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{4}{3}e^{-t} \end{bmatrix}$$

check: $\Phi(t_0) = I$? yes

another way to find $\Phi(t)$

$$\vec{x}\,' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x} \qquad \vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

use the fact $\Phi(t_0) = I$

find c_1 and c_2 such that $\vec{x}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (first column of $\Phi(t_0)$)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 1 \end{array}\right]$$

$$c_2 = -\frac{1}{3} \qquad c_1 = \frac{2}{3}$$

$\vec{x}(t)$ using these will become <u>first column</u> of $\Phi(t)$

then repeat with $\vec{x}(t_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (2nd col. of $\Phi(t_0)$)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 0 \end{array}\right] \quad \ldots \quad c_1 = -1/3 \quad c_2 = 2/3$$

$\vec{x}(t)$ using these becomes <u>second column</u> of $\Phi(t)$

this is better method if A is 3×3 or bigger

Connection between $y' = ay$ and $\vec{x}' = A\vec{x}$

$y' = ay$ has solution $\quad y = ce^{at} \quad y(0) = c$

$$y = e^{at} \cdot y(0)$$

$\vec{x}' = A\vec{x}$ has solution $\quad \vec{x}(t) = \Phi(t)\vec{x}(0)$

so $\boxed{\Phi(t) = e^{At}}$ matrix exponential