7.8 Repeated Eigenvalues

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)^2 = 0 \qquad \lambda = 1, 1$$

$\lambda = 1$ has algebraic multiplicity of two

$\underline{\lambda = 1}$ solve $(A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_1 = r \\ v_2 = s \end{array}$$

r, s are arbitray constants

$$\vec{v} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

note $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent

so there are two eigenvectors $\rightarrow$ $\lambda = 1$ has geometric multiplicity of two solution: $\vec{X}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

when algebraic ~~multipl~~ multiplicity = geometric multiplicity

there is enough eigenvectors to form solution

$\rightarrow$ matrix A is <u>complete</u>

<u>example</u> $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

$\lambda = 1, 1$ algebraic mult. = 2

$\underline{\lambda = 1}$ $(A - \lambda I)\vec{v} = \vec{0}$

$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right]$ only <u>one</u> free variable

$v_2 = r$ $v_1 = 2r$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ only <u>one</u> vector

geometric mult. = 1

matrix A is <u>defective</u>

$\vec{x}_1 = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad \vec{x}_2 = ?$

revisit scalar case : $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0 \qquad r = -2, -2$

$y = c_1 e^{-2t} + c_2 \underline{t} e^{-2t}$

this suggests we might be able to use

$\vec{x}_2 = \underline{t} e^{\lambda t} \vec{v}$

check: it must satisfy $\vec{x}\,' = A\vec{x}$

$$\underbrace{(t\lambda e^{\lambda t} + e^{\lambda t})\vec{v}}_{\vec{x}_2\,'} = Ate^{\lambda t}\vec{v}$$

terms w/ $te^{\lambda t}$: $\lambda \vec{v} = A\vec{v}$ $\rightarrow$ λ is eigenvalue
$\vec{v}$ is eigenvector
(of course...)

terms w/ $e^{\lambda t}$: $\vec{v} = \vec{0}$ $\rightarrow$ eigenvector is $\vec{0}$
but this contradicts
$\det(A-\lambda I)=0$

this means the <u>form</u> of $\vec{x}_2$ is <u>wrong</u>
↳ $\vec{x}_2 = te^{\lambda t}\vec{v}$

must modify form: need an $e^{\lambda t}$ term on the right side of $\vec{x}' = A\vec{x}$

new form: $$\boxed{\vec{x}_2 = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}}$$

eigenvector

generalized eigenvector

plug into $\vec{x}' = A\vec{x}$

$$(t\lambda e^{\lambda t} + e^{\lambda t})\vec{v} + \lambda e^{\lambda t}\vec{u} = Ate^{\lambda t}\vec{v} + Ae^{\lambda t}\vec{u}$$

$te^{\lambda t}$ terms: $\lambda\vec{v} = A\vec{v}$

$e^{\lambda t}$ terms: $\vec{v} + \lambda\vec{u} = A\vec{u}$

$$A\vec{u} - \lambda\vec{u} = \vec{v}$$

$$\boxed{(A - \lambda I)\vec{u} = \vec{v}}$$

solve this for $\vec{u}$

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back to $\vec{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\vec{x}$

$\lambda = 1, 1$ $\quad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \vec{x}_1 = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\vec{x}_2 = te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t\vec{u}$$

find $\vec{u}$: $(A-\lambda I)\vec{u} = \vec{v}$

$$\left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$u_2 = r$

$u_1 = 1 + 2r$

$$\vec{u} = \underbrace{\cancel{r\begin{bmatrix} 2 \\ 1 \end{bmatrix}}}_{\substack{\text{not linearly} \\ \text{independent} \\ \text{from } \vec{v} \\ \text{(discard,} \\ \text{choose } r=0)}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\boxed{\vec{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left\{ te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}}$

what if ~~to~~ we let $u_1 = r$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} u_1 &= r \\ r - 2u_2 &= 1 \end{aligned} \quad \longrightarrow \quad u_2 = \frac{r-1}{2} = \frac{1}{2}r - \frac{1}{2}$$

$$\cancel{u_2 = \frac{-1+r}{2} = -\frac{1}{2} + \frac{1}{2}r}$$

so $$\vec{u} = \underbrace{\cancel{r\begin{bmatrix} 1 \\ +\frac{1}{2} \end{bmatrix}}}_{\text{same as } \vec{v} \text{ (discard)}} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Solution: } \vec{X}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left\{ te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right.}$$

Same solution but look different (because c_1, c_2 are <u>different</u>)

Suppose $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

first form: $\vec{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left\{ te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$c_1 = 0$ (2nd row)

$c_2 = 2$

$$\vec{x}(t) = \begin{bmatrix} 4te^t + 2e^t \\ 2te^t \end{bmatrix}$$

2nd form: $\vec{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left\{ te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$$

$c_1 = 1$

$c_2 = 2$

$$\vec{x}(t) = \begin{bmatrix} 2e^t + 4te^t \\ 2te^t \end{bmatrix}$$