7.9 Nonhomogeneous Systems

$$\vec{x}\,' = A\vec{x} + \vec{g}(t)$$

several methods in the book

diagonalization – decouple equations and solve first-order eqs.

in class $\left\{ \begin{array}{l} \text{undetermined coefficients – depends on form of } \vec{g}(t) \\ \text{variation of parameters – based on fundamental matrix } \Psi \text{, most general} \end{array} \right.$

Laplace transform – needs initial condition

example $\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$

Solve the homogeneous part first, regardless of method. $\vec{x}' = A\vec{x}$

$$\vec{x}_h = c_1 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

solution: $\vec{x} = \vec{x}_h + \vec{x}_p$

↪ due to $\vec{g}(t)$

undetermined coefficients

guess an $\vec{x}_p$, solve coefficients

here, $\vec{x}_p = \vec{a}\, e^t$

↪ vector constant

plug into $\vec{x}' = A\vec{x} + \vec{g}$

$$\underbrace{\vec{a}e^t}_{\vec{x}_p{}'} = A\underbrace{\vec{a}e^t}_{\vec{x}_p} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^t$$

e^t terms: $\vec{a} = A\vec{a} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$(I - A)\vec{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 4 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 6 & -5 \end{array}\right]$$

$a_2 = -5/6$

$a_1 = 7/6$

$\vec{a} = \frac{1}{6}\begin{bmatrix} 7 \\ -5 \end{bmatrix}$

general solution: $\vec{x} = \underbrace{c_1 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\frac{1}{6}\begin{bmatrix} 7 \\ -5 \end{bmatrix}e^t}_{\vec{x}_p}$

remember undetermined coefficients is only usable

if $\vec{g}(t)$ contains: polynomials

exponentials

trig.

for example, $\vec{g}(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} t^{-3}$

NOT a polynomial

undet. coeff. <u>cannot</u>

be used for this

also, be careful when $\vec{g}(t)$ contains terms already in $\vec{x}_h$

$\cancel{\vec{x}(t)}$

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-2t}$$

$$\vec{x}_h = c_1 e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{x}_p = \vec{a}\,t\,e^{-2t} + \underbrace{\vec{b}\,e^{-2t}}_{\text{must add this in vector case}}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \vec{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} t}_{\text{first deg. polynomial}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}}_{\text{exponential (duplicates } \vec{x}_h)}$$

$$\vec{x}_p = \underbrace{\vec{a}\, t + \vec{b}}_{\text{for the first deg. polynomial}} + \underbrace{\vec{c}\, t e^{-2t} + \vec{d}\, e^{-2t}}_{\text{for the } e^{-2t} \text{ part}}$$

Variation of Parameters

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

$\vec{x}_h = \Psi(t)\,\vec{c}$ vector of constants ("parameters")

general solution: $\vec{x} = \Psi(t)\,\underbrace{\vec{u}(t)}$

↳ varied parameters

plug into DE: $\vec{x}' = \Psi(t)\,\vec{u}'(t) + \Psi'(t)\,\vec{u}(t)$

$$\Psi\vec{u}' + \Psi'\vec{u} = A\Psi\vec{u} + \vec{g}$$

terms w/ $\vec{u}$: $\Psi' = A\Psi$ $\longrightarrow$ analogous to $\vec{x}' = A\vec{x}$

~~terms w/o~~ ~~w/~~ $\cancel{\vec{u}'}$:

terms w/o $\vec{u}$: $\boxed{\Psi\vec{u}' = \vec{g}}$ solve for $\vec{u}'$

$$\boxed{\vec{u}\,' = \Psi^{-1}\vec{g}}$$ solve for $\vec{u}\,'$

<u>example</u>

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$$

$$\Psi = \begin{bmatrix} -e^{-2t} & -e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$$

$$\vec{x} = \Psi\vec{u} \quad \text{where} \quad \Psi\vec{u}\,' = \vec{g}$$

$$\left[\begin{array}{cc|c} -e^{-2t} & -e^{-t} & 2e^{t} \\ 2e^{-2t} & e^{-t} & -e^{t} \end{array}\right]$$

$$\left[\begin{array}{cc|c} -e^{-2t} & -e^{-t} & 2e^{t} \\ 0 & -e^{-t} & 3e^{t} \end{array}\right]$$

$$-e^{-t}u_2' = 3e^{t}$$

$$u_2' = -3e^{2t}$$

$$u_1' = \frac{e^{-t}u_2' + 2e^{t}}{-e^{-2t}} = -e^{t}u_2' - 2e^{3t}$$

$$= 3e^{3t} - 2e^{3t} = e^{3t}$$

$$u_1 = \frac{1}{3}e^{3t} + c_1 \qquad \vec{x} = \Psi\vec{u}$$

$$u_2 = -\frac{3}{2}e^{2t} + c_2$$

$$\vec{x} = \begin{bmatrix} -e^{-2t} & -e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{3}e^{3t} + c_1 \\ -\frac{3}{2}e^{2t} + c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}e^{t} - c_1e^{-2t} + \frac{3}{2}e^{t} - c_2e^{-t} \\ \frac{2}{3}e^{t} + 2c_1e^{-2t} - \frac{3}{2}e^{t} + c_2e^{-t} \end{bmatrix}$$

$$= c_1e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 7/6 \\ -5/6 \end{bmatrix}e^{t}$$

Quiz 6

1. Solve

$$\boldsymbol{x}' = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \boldsymbol{x}$$

2. The system

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}$$

Has eigenvalues 3 and -1 and corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Find the fundamental matrix $\Phi(t)$ such that $\Phi(0) = I$