5.2 Series Solutions near an Ordinary Point (part 1)

linear DE: $P(x)y'' + Q(x)y' + R(x)y = 0$

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

if $P(x) \neq 0$, then that x is an ordinary point

want solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

example $y'' + xy' + y = 0$

find series solution near $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

due to loss of constant a_0

$$y'' = \sum_{n=2}^{\infty} a_n \cdot (n)(n-1) x^{n-2}$$

sub into DE

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + x \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n \cdot n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

shift

want these to match

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_n \cdot n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

when $n=0$: $2a_2 + a_0 = 0 \rightarrow a_2 = -\frac{1}{2}a_0$

when $n \geq 1$: $\underbrace{\left[a_{n+2}(n+2)(n+1) + n \cdot a_n + a_n\right]}_{\text{zero}} x^n = 0$

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$$

$$a_{n+2} = -\frac{(n+1)a_n}{(n+2)(n+1)}$$

$$\boxed{a_{n+2} = \frac{-a_n}{n+2} \qquad n = 1, 2, 3, 4, \ldots}$$

recurrence relation

a_{n+2} is related to a_n

so a_2 to a_0, a_4 to a_2, ...

a_3 to a_1, a_5 to a_3, ...

all even n's are related

all odd n's are related

if n is <u>even</u> : $n = 2, 4, 6, 8, \ldots$

$$a_2 = -\frac{1}{2} a_0$$

$n=2$: $a_4 = -\frac{a_2}{4} = -\frac{1}{4}\left(-\frac{1}{2}a_0\right) = \frac{1}{8} a_0$

$n=4$: $a_6 = -\frac{a_4}{6} = -\frac{1}{6}\left(\frac{1}{8}a_0\right) = -\frac{1}{48} a_0$

$n=6$: $a_8 = -\frac{a_6}{8} = -\frac{1}{8}\left(-\frac{1}{48}a_0\right) = \frac{1}{384} a_0$

if n is <u>odd</u> : $n = 1, 3, 5, 7, \ldots$

$n=1$: $a_3 = -\frac{a_1}{3}$

$n=3$: $a_5 = -\frac{a_3}{5} = -\frac{1}{5}\left(-\frac{a_1}{3}\right) = \frac{1}{15} a_1$

$n=5$: $a_7 = -\frac{a_5}{7} = -\frac{1}{7}\left(\frac{1}{15}a_1\right) = -\frac{1}{105} a_1$

$n=7$: $a_9 = -\frac{a_7}{9} = -\frac{1}{9}\left(-\frac{1}{105}a_1\right) = \frac{1}{945} a_1$

Solution: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$\cancel{y = a_0 + a_1 x + \underbrace{\frac{1}{8}}_{a_2} a_0 x^2 + \underbrace{\frac{1}{15}}_{a_3} a_1 x^3 - \frac{1}{48} a_0 x^4 - \frac{1}{105} a_1 x^5 + \cdots}$$

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 - \frac{1}{48} a_0 x^6 - \frac{1}{105} a_1 x^7 + \cdots$$

$$= a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6 + \cdots\right) + a_1 \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5 - \frac{1}{105} x^7 + \cdots\right)$$

$$y = a_0 y_1 + a_1 y_2$$

y_1, y_2: linearly independent fundamental solutions

a_0, a_1: constants (depend on initial conditions)

compare to

$y'' - y = 0$

$y = c_1 e^x + c_2 e^{-x}$

Suppose we know $y(0) = 0$ and $y'(0) = 1$

imply $x_0 = 0$

Solution from last page:

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots\right)$$
$$+ a_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \cdots\right)$$

$$y(0) = a_0 = y(0) = 0 \longrightarrow \boxed{a_0 = y(x_0)}$$

$$y' = a_0\left(-x + \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots\right)$$
$$+ a_1\left(1 - x^2 + \frac{1}{3}x^4 - \frac{7}{105}x^6 + \cdots\right)$$

$$y'(0) = a_1 = 1 \longrightarrow \boxed{a_1 = y'(x_0)}$$

__example__ $y'' + xy' + y = 0$ $\qquad$ $x_0 = 1$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} a_n (n)(x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1)(x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)(x-1)^{n-2} + x \sum_{n=1}^{\infty} a_n(n)(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

rewrite x as : $\boxed{1 + (x-1)}$ Taylor series of x near $x = 1$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)(x-1)^{n-2} + [1+(x-1)] \sum_{n=1}^{\infty} a_n(n)(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$$\sum_{n=1}^{\infty} a_n(n)(x-1)^{n-1} + \sum_{n=1}^{\infty} a_n(n)(x-1)^n$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n$$

$$+ \sum_{n=1}^{\infty} a_n(n)(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

$n=0$: $\quad a_2(2)(1) + a_1(1) + a_0 = 0$

$$a_2 = \frac{-a_0 - a_1}{2}$$

a_0, a_1 independent and relate to initial conditions

$n \geq 1$: $\quad (n+2)(n+1)a_{n+2} + (n+1)a_{n+1}$

$$+ n a_n + a_n = 0$$

$$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n = 0$$

$$(n+2)(n+1)a_{n+2} = -(n+1)a_{n+1} - (n+1)a_n$$

$$\boxed{a_{n+2} = \frac{-a_{n+1} - a_n}{n+2} \qquad n = 1, 2, 3, 4, \ldots}$$

recurrence relation