8.1 + 8.2 Numerical Integration

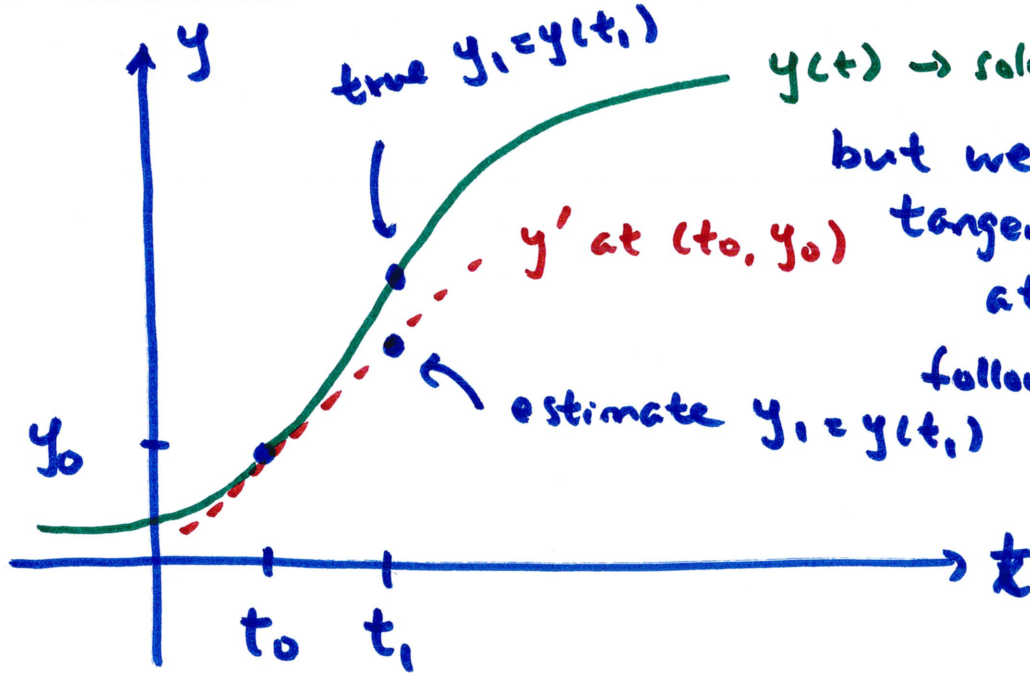
Euler's (or Tangent Line) Method

Improved Euler (or Heun's) Method

used when integration is impossible or impractical
(also how things like Matlab integrate DE's)

Euler's Method

$y' = f(t,y) \quad y(t_0) = y_0$



but we know the tangent line slope y' at every (t, y)

follow tangent line

tangent line eq. at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)t$$

take a small step h ("step size") to go to t_1

$(t_1 = t_0 + h)$

estimate of $y_1 = y(t_1)$ from tangent line is

$$y_1 \approx y_0 + f(t_0, y_0)h$$

then move by h to go to t_2

$$y_2 \approx y_1 + f(t_1, y_1)h$$

$$\vdots$$

$$\boxed{y_{n+1} \approx y_n + \underbrace{f(t_n, y_n)}_{\text{slope at old } y} h}$$

Euler Method algorithm

new y (estimated)

old y

step size

<u>example</u> $y' = 2y - 3t \qquad y(0) = 1$

use $h = 0.05$ to estimate $y(0.1)$

$y_0 = 1 \qquad t_0 = 0$

$t_1 = t_0 + h = 0.05$

$y_1 = y(0.05) \approx y_0 + f(t_0, y_0)h$

$\approx 1 + (2 \cdot \underset{y_0}{\underset{\uparrow}{1}} - 3 \cdot \underset{t_0}{\underset{\uparrow}{0}})(0.05) \approx 1.1$

$t_2 = t_1 + h = 0.1$

$y_2 = y(0.1) \approx y_1 + f(t_1, y_1)h$

$\approx 1.1 + (2 \cdot \underbrace{1.1}_{y_1} - 3 \cdot \underbrace{0.05}_{t_1})(0.05) \approx \boxed{1.2025}$

estimated $y(0.1)$

$y' = 2y - 3t \quad y(0) = 1$ can actually be solved by hand
(usually not the case)

$y' - 2y = -3t$ integrating factor $I = e^{\int -2\,dt} = e^{-2t}$

$$e^{-2t}y' - 2e^{-2t}y = -3te^{-2t}$$

$$\frac{d}{dt}\left(e^{-2t}y\right) = -3te^{-2t}$$

$$e^{-2t}y = \int -3te^{-2t}\,dt = \frac{3}{4}e^{-2t}(2t+1) + C$$

$$y = \frac{3}{4}(2t+1) + Ce^{2t}$$

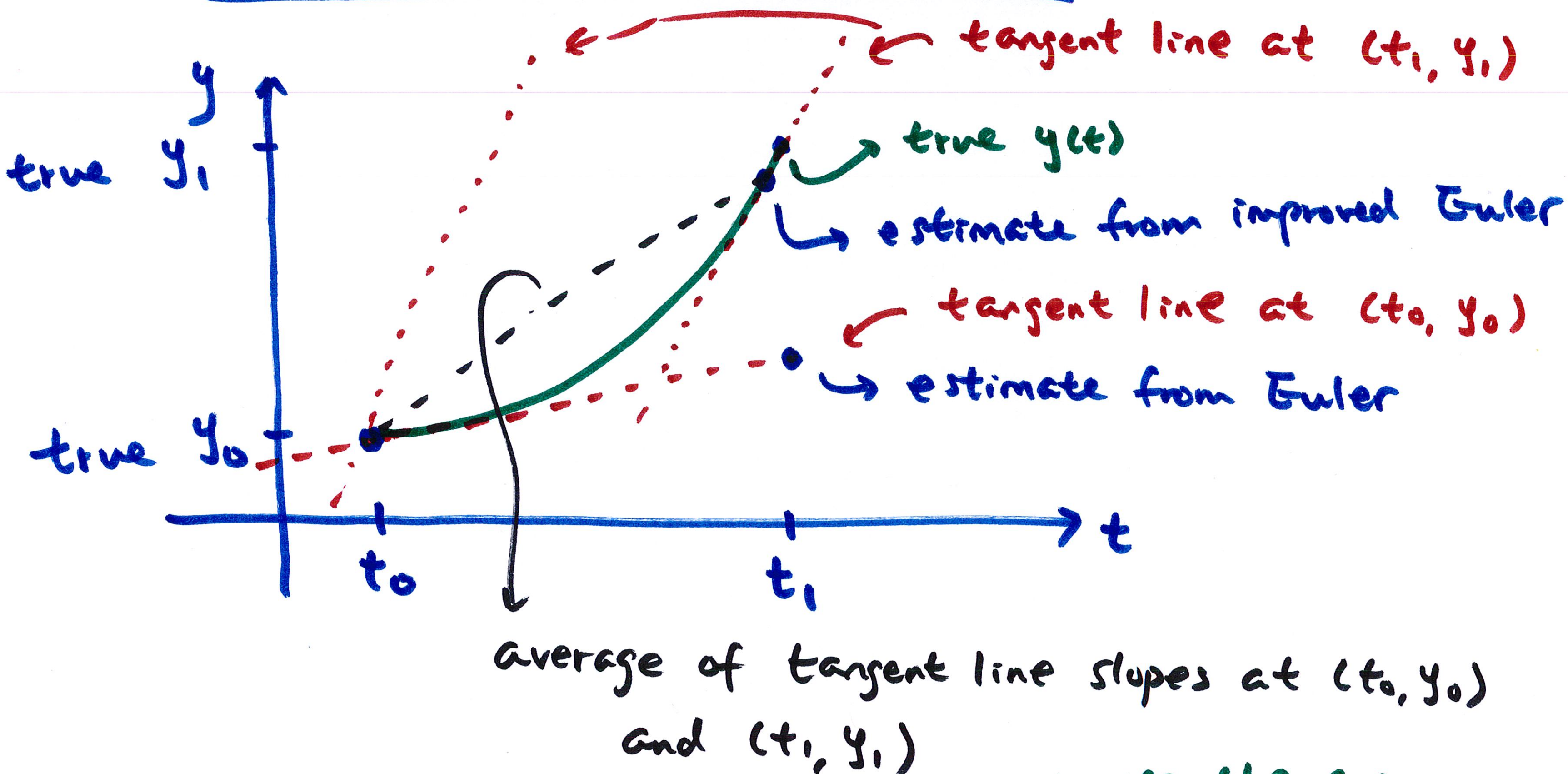
$$1 = \frac{3}{4} + C \quad \rightarrow C = \frac{1}{4}$$

$$y = \frac{3}{4}(2t+1) + \frac{1}{4}e^{2t}$$

<u>true</u> $y(0.1) = 1.20535$ Euler : 1.2025

Euler w/ $h = 0.025$: 1.20388

Improved Euler / Heun's Method



Euler: $y_{n+1} \approx y_n + \boxed{f(t_n, y_n)}\, h$

use the average slope instead

don't know this

Heun's: $y_{n+1} \approx y_n + \frac{1}{2}\Big[\underbrace{f(t_n, y_n)}_{\text{slope at start}} + \underbrace{f(t_{n+1}, \boxed{y_{n+1}})}_{\text{slope at end}}\Big]\, h$

but we don't know y_{n+1} on right side to

find $f(t_{n+1}, y_{n+1})$

so we use (original) Euler to estimate y_{n+1} on right

Heun's :

$$y_{n+1} \approx y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + f(t_n, y_n)h)\right]h$$

in practice, first <u>predict</u> y_{n+1} using Euler

then <u>correct</u> y_{n+1} using Heun's

(this is an example of a <u>predictor-corrector</u> method)

example $y' = 2y - 3t \quad y(0) = 1$

$h = 0.05$, find $y(0.1)$

predict $y(0.05)$: $y_1^* = y_0 + f(t_0, y_0)h$

$= 1 + (2\cdot 1 - 3\cdot 0)(0.05) = 1.1$

correct it : $y_1 = 1 + \frac{1}{2}\left[(2\cdot 1 - 3\cdot 0) + (2\cdot \underbrace{1.1}_{y_1^*} - 3\cdot \underbrace{0.05}_{t_1})\right](0.05)$

≈ 1.10125

predict $y(0.1)$: $y_2^* = y_1 + f(t_1, y_1)h$

$= 1.10125 + (2\cdot 1.10125 - 3\cdot 0.05)(0.05)$

$= 1.203875$

correct it : $y_2 = 1.10125 + \frac{1}{2}\left[(2\cdot 1.10125 - 3\cdot 0.05)\right.$

$\left. + (2\cdot 1.203875 - 3\cdot 0.1)\right](0.05)$

$y(0.1) = y_2 \approx 1.20525625$ Euler : 1.2025

True : 1.20535

higher order eqs?

$$y'' + y = 0 \qquad y(0) = 1, \quad y'(0) = 0$$

Convert to system of 1st order eqs.

$$x_1 = y$$

$$x_2 = y'$$

$$x_1' = x_2 = f_1(x_1, x_2, t)$$

$$x_2' = -x_1 = f_2(x_1, x_2, t)$$

use $h = 0.05$ as example

$$x_1(0) = 1 \qquad (IC)$$

$$x_1(0.05) \approx x_1(0) + f_1(x_1(0), x_2(0), 0)(0.05)$$

$$\approx 1 + 0(0.05) = 1$$

$$x_2(0.05) \approx x_2(0) + f_2(x_1(0), x_2(0), 0)(0.05)$$

$$\approx -0.05$$