10.1 Two-Point Boundary Value Problems

exam 2 covers up to this section

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

instead of initial conditions $y(t_0) = y_0$ $y'(t_0) = y_1$

we now specify <u>boundary</u> conditions

$$y(\alpha) = y_0 \quad \text{and} \quad y(\beta) = y_1$$

$$\text{or } y'(\alpha) = y_0' \qquad \text{or } y'(\beta) = y_1'$$

conditions are at <u>ends</u> of the interval $\alpha \le x \le \beta$

if $g(t)$ and ~~y_0~~ both boundary conditions are all zero, then the problem is homogeneous.

homogeneous boundary value problems, like homogeneous system $A\vec{x} = \vec{0}$ always have the trivial solution

$\rightarrow$ we usually don't want this solution

nonhomogeneous boundary value problems, like $A\vec{x} = \vec{b}$

can have : no solution

unique solution

infinitely many solutions

<u>example</u> $y'' + 2y = 0 \quad y'(0) = 1, \; y'(\pi) = 0$ nonhomogeneous

characteristic eq: $r^2 + 2 = 0 \quad r = \pm\sqrt{2}\, i$

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)$$

$$y' = -\sqrt{2}\, c_1 \sin(\sqrt{2}x) + \sqrt{2}\, c_2 \cos(\sqrt{2}x)$$

apply BC's:

$$1 = \sqrt{2}\, c_2 \cos(0) \rightarrow c_2 = \frac{1}{\sqrt{2}}$$

$$0 = -\sqrt{2}\, c_1 \sin(\sqrt{2}\pi) + \frac{\sqrt{2}}{\sqrt{2}} \cos(\sqrt{2}\pi)$$

$$c_1 = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi)$$

solution: $\boxed{y = \frac{1}{\sqrt{2}} \cot(\sqrt{2}\pi) \cos(\sqrt{2}x) + \frac{1}{\sqrt{2}} \sin(\sqrt{2}x)}$

(unique solution)

example $y'' + y = 0 \qquad y'(0) = 1 \qquad y'(\pi) = 0$

$$r^2 + 1 = 0 \quad \rightarrow \quad r = \pm i$$

$$y = C_1 \cos x + C_2 \sin x$$

$$y' = -C_1 \sin x + C_2 \cos x$$

$$\left.\begin{aligned} 1 &= C_2 \\ 0 &= -C_2 \end{aligned}\right\} \text{ inconsistent} \qquad \text{no solution}$$

example $y'' + 4y = \sin x \qquad y(0)=0,\ y(\pi)=0$

solve $y''+4y=0$ first $\qquad r^2+4=0 \quad r=\pm 2i$

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = A\cos x + B\sin x$$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x$$

plug into $y''+4y=\sin x$

$$-A\cos x - B\sin x + 4A\cos x + 4B\sin x = \sin x$$

$$A=0 \qquad B = 1/3$$

$$y = C_1\cos 2x + C_2 \sin 2x + \tfrac{1}{3}\sin x$$

$y(0)=0$: $\quad 0 = C_1$

$y(\pi)=0$: $\quad 0 = C_2 \sin(2\pi) + \tfrac{1}{3}\sin(\pi) = 0$

C_2 is arbitrary

solution: $y = C_2 \sin 2x + \frac{1}{3} \sin x$ C_2 = arbitrary number

infinitely many solutions

Eigenvalue Problem

$$y'' + \lambda y = 0 \qquad y(0) = y(\pi) = 0$$

what does λ have to be to have nontrivial solutions?

λ : some real number $\begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases}$

$\underline{\lambda > 0}$ let $\lambda = \mu^2$ (to avoid having many $\sqrt{\ }$ in next steps)

$$y'' + \mu^2 y = 0 \qquad y(0) = y(\pi) = 0$$

$$y = C_1 \cos \mu x + C_2 \sin \mu x$$

gives trivial

apply BC's: $\quad 0 = C_1$

$$0 = C_2 \sin(\mu\pi) \rightarrow \cancel{C_2 = 0} \text{ or } \sin\mu\pi = 0$$

$$\sin\mu\pi = 0 \rightarrow \mu\pi = n\pi \qquad n = 0, 1, 2, 3, \ldots$$

$$\mu = 0, 1, 2, 3, \ldots$$

and since $\lambda = \mu^2 = \cancel{0}, 1, 4, 9, 16, 25, \ldots$

because we are looking at $\lambda > 0$

these λ's that produce nontrivial solutions are called <u>eigenvalues</u>

solution: $\quad y = C_2 \sin\sqrt{\lambda}\, x \qquad \lambda = 1, 4, 9, 16, 25, \ldots$

all nontrivial solutions (for $\lambda > 0$)

are in the form $\underbrace{\sin\sqrt{\lambda}\, x}$

eigenfunction

$\lambda = 0$ $\quad y'' + \lambda y = 0 \qquad y(0) = y(\pi) = 0$

$y'' = 0 \qquad y = c_1 x + c_2$

apply BC's: $\quad 0 = c_2$

$0 = c_1 \cdot \pi \rightarrow c_1 = 0$

but this means $y = 0$ (trivial)

$\lambda = 0$ is NOT an eigenvalue

$\underline{\lambda < 0}$ $y'' + \lambda y = 0$ $y(0) = y(\pi) = 0$

let $\lambda = -\mu^2$ $r^2 - \mu^2 = 0$ $r = \pm\mu$

$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$

apply BC's:

$0 = c_1 + c_2 \rightarrow c_2 = -c_1$

$0 = c_1 e^{\mu\pi} + c_2 e^{-\mu\pi}$

$0 = c_1 (e^{\mu\pi} - e^{-\mu\pi}) \rightarrow c_1 = 0,\ c_2 = 0$

trivial solution

$y'' + \lambda y = 0$ $y(0) = y(\pi) = 0$

has only positive eigenvalues $\lambda = 1, 4, 9, 16, 25, \ldots$

and eigenfunctions $\sin\sqrt{\lambda}\, x$

example $y'' + \lambda y = 0$ $\quad y'(0) = \cancel{y'(L)} = 0$

$= y'(L) = 0$

$\lambda > 0$ $\quad \lambda = \mu^2$

$$y = c_1 \cos \mu x + c_2 \sin \mu x$$

$$y' = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$$

$0 = \mu c_2$ $\quad \mu \neq 0$ because $\lambda = \mu^2 > 0$

so $c_2 = 0$

$0 = -\mu c_1 \sin \mu L$ $\quad c_1 \neq 0$ (otherwise trivial)

$\sin \mu L = 0$

$\mu L = n\pi$ $\quad n = 1, 2, 3, \ldots$

$\mu = \frac{n\pi}{L}$

$\boxed{\lambda = \mu^2 = \left(\frac{n\pi}{L}\right)^2}$ eigenvalues $n = 1, 2, 3, \ldots$

Solution: $y = c_1 \cos\left(\frac{n\pi x}{L}\right)$

$$\boxed{\text{eigenfunctions: } \cos\left(\frac{n\pi x}{L}\right)}$$

$\underline{\lambda = 0}$ $\quad y'' + \lambda y = 0 \qquad y'(0) = y'(L) = 0$

$y'' = 0 \rightarrow y = c_1 x + c_2$

$y' = c_1$

$$\left.\begin{array}{ll} y'(0) = 0 : & 0 = c_1 \\ y'(L) = 0 : & 0 = c_1 \end{array}\right\} \quad c_2 \text{ is arbitrary (but not 0)}$$

solution: $y = c_2 = c_2 \cdot 1$

$$\boxed{\begin{array}{l} \lambda = 0 \text{ is an eigenvalue} \\ \text{eigenfunction is } 1 \end{array}}$$

$\underline{\lambda < 0}$ $\quad y'' + \lambda y = 0 \qquad y'(0) = y'(L) = 0$

$$\lambda = -\mu^2$$

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$y' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x}$$

$y'(0)=0: \quad 0 = \mu c_1 - \mu c_2 \rightarrow c_1 = c_2$

$y'(L)=0: \quad 0 = \mu c_1 e^{\mu L} - \mu c_2 e^{-\mu L}$

$$0 = \mu c_1 (e^{\mu L} - e^{-\mu L}) \rightarrow c_1 = 0$$

$$c_2 = 0$$

(trivial)

no negative eigenvalues