10.2 Fourier Series

NOT on exam 2

Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

can express analytic functions as a polynomial

near $x = x_0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Fourier Series: $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

can express periodic functions as sum of

sines and cosines.

L: half of the fundamental period of f(x)

sin x has period of $2\pi, 4\pi, 6\pi, 8\pi, \ldots$

Smallest is the fundamental period

the constants a_0, a_n, b_n can be found by the Euler-Fourier Formulas

derivation in book

given on exam and quizzes

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

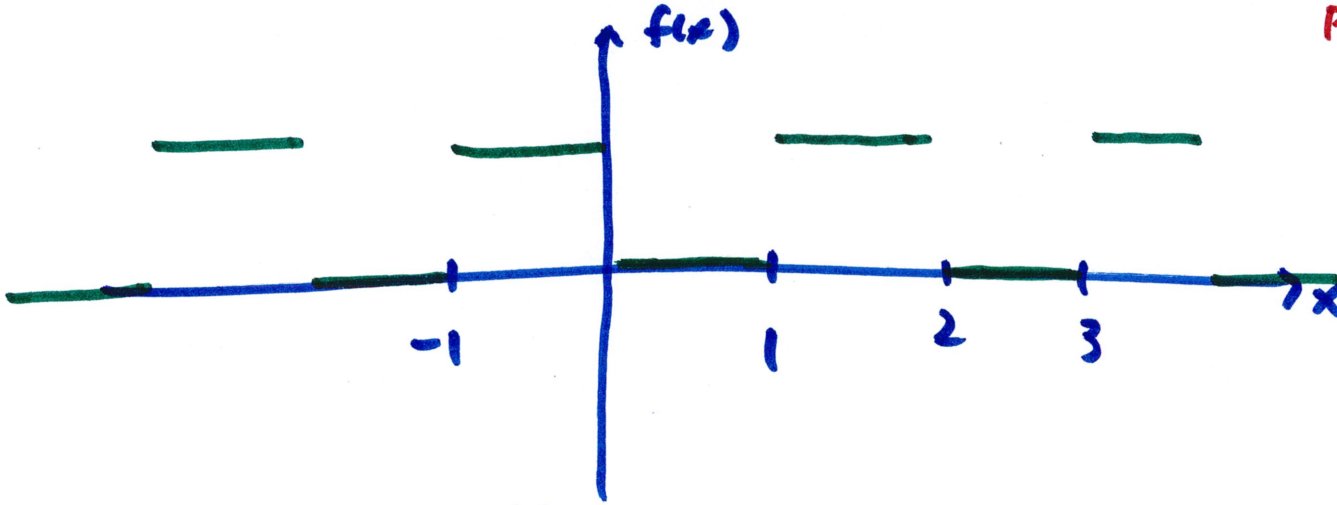
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

example express $f(x) = \begin{cases} 1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \end{cases}$ $f(x+2) = f(x)$

as a Fourier series.

$\underset{\text{fundamental period}}{\underbrace{2}}$



$L = 1$ (half of fund. period)

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} \cos n\pi x\,dx = \frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{0}$$

$$= \frac{1}{n\pi}\Big(0 - \sin(-n\pi)\Big) = \frac{1}{n\pi}\cdot\sin(n\pi) \qquad n=1,2,3,\ldots$$

$$= 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx = -\frac{1}{n\pi}\cos n\pi x\Big|_{-1}^{0}$$

$$= -\frac{1}{n\pi}\left(1 - \cos(-n\pi)\right) = -\frac{1}{n\pi}\left(1 - \cos n\pi\right)$$

$n = 1, 2, 3, \ldots$

$$\cos(n\pi) = (-1)^n$$

$$= -\frac{1}{n\pi}\left(1 - (-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= \frac{1}{2} + \sum_{n=1,3,5,7..}^{\infty} -\frac{2}{n\pi}\sin(n\pi x)$$

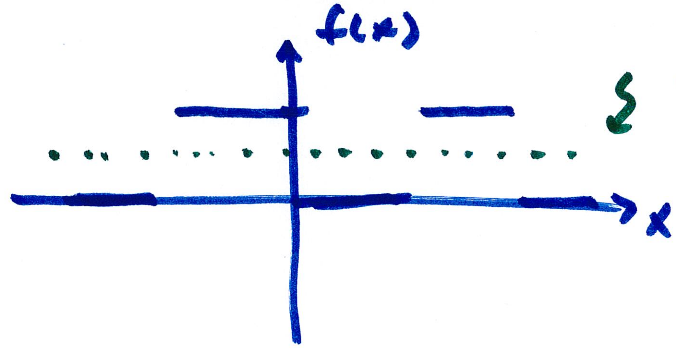
better way to write this

$n = 2k - 1 \qquad k = 1, 2, 3, \ldots$

$$\boxed{f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} -\frac{2}{(2k-1)\pi} \sin\left[(2k-1)\pi x\right]}$$

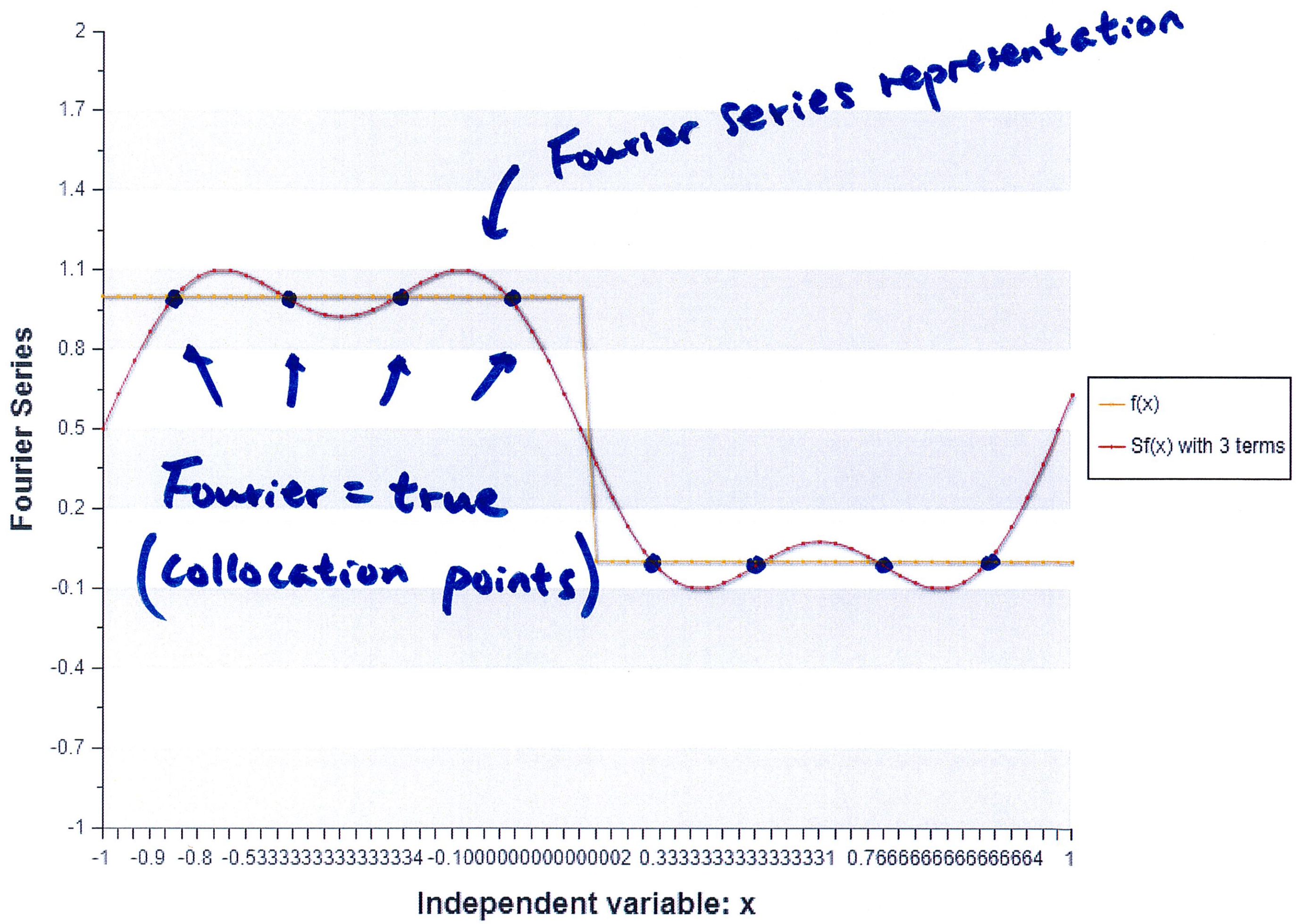
converges to $f(x) = \begin{cases} 1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \end{cases}$ $\quad f(x+2) = f(x)$

as $k \to \infty$



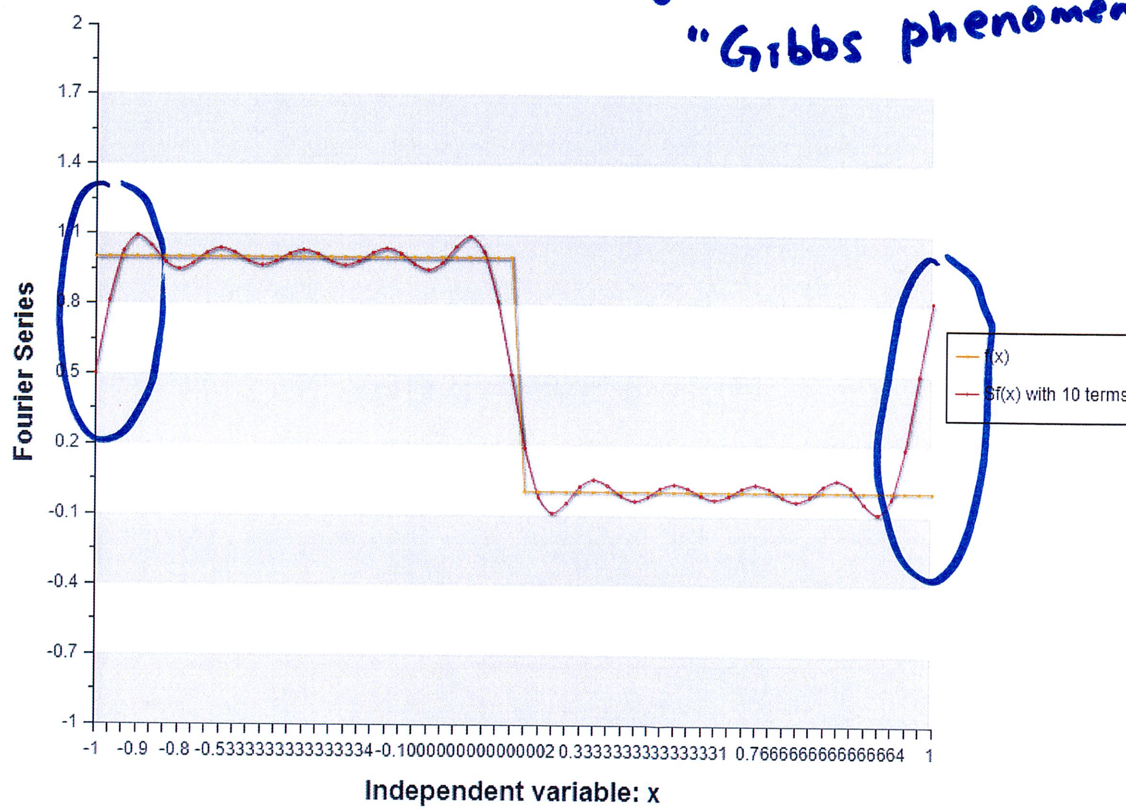
$\} \frac{1}{2}$ (const. term)

n	an	bn
0	0.5000018823344495	0
1	0	-0.636619704879964
2	-0.0000017314464604978381	-0.000004333398193366315
3	-0.0000019252781029012164	-0.2122098806338695

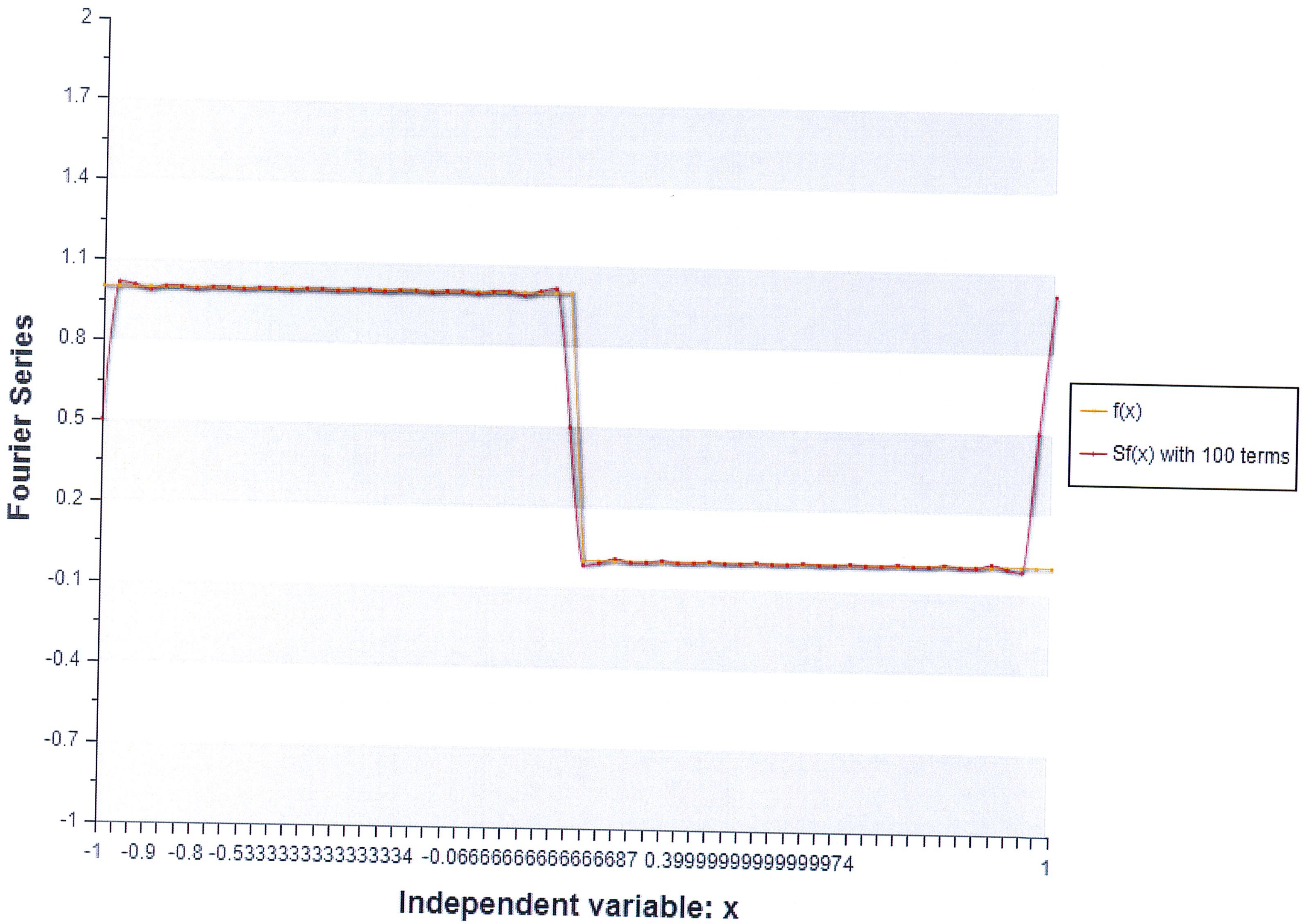


n	an	bn
0	0.5000018823344495	0
1	0	-0.636619704879964
2	-0.0000017314464604978381	-0.000004333398193366315
3	-0.0000019252781029012164	-0.2122098806338695
4	0.0000033534222248911488	0
5	0	-0.1273247844868135
6	0.0000028621728033339665	0
7	0	-0.09094425589104733
8	0	-0.000001306403214505632
9	-0.000002041347100419803	-0.07073493354288676
10	0.0000010593624098675818	-0.0000024169021346840926

large errors at ends
"Gibbs phenomenon"



n=100



Fourier Series
2
1.7
1.4
1.1
0.8
0.5
0.2
-0.1
-0.4
-0.7
-1
-1 -0.9 -0.8 -0.5333333333333334 -0.06666666666666687 0.39999999999999974 1
Independent variable: x
f(x)
Sf(x) with 100 terms

example

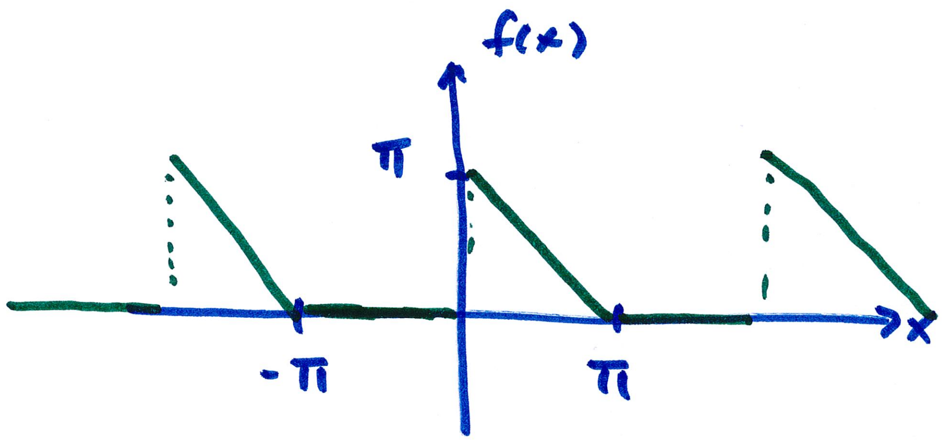
$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$$

$f(x + 2\pi) = f(x)$

$\downarrow$

$2L$: fund. period

$L = \pi$



$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)\,dx = \frac{1}{\pi} \left(\pi x - \frac{x^2}{2} \right) \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \frac{\pi}{2}$$

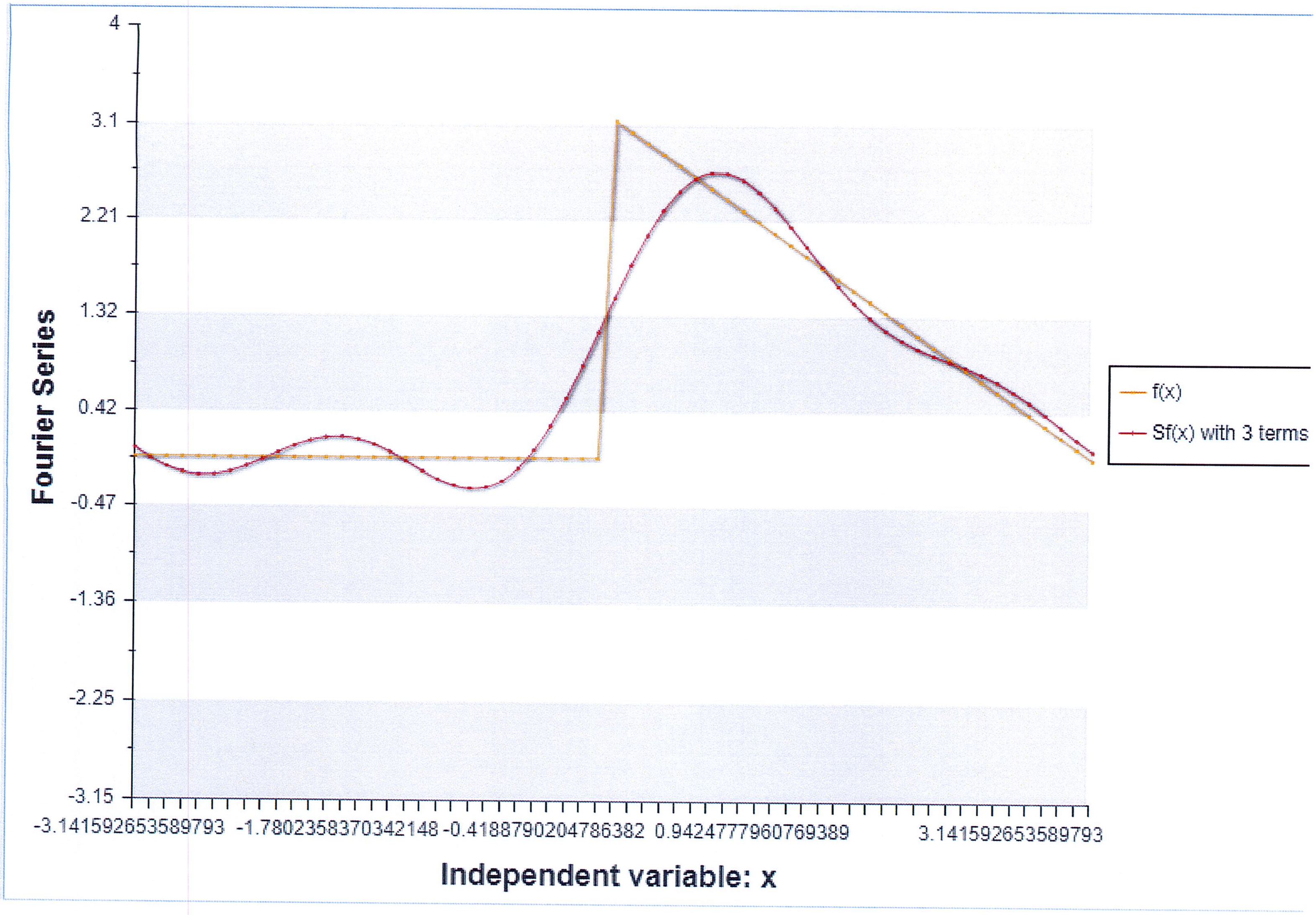
$$a_n = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx \qquad \text{by parts}$$

$n = 1, 2, 3, \dots$

$$= \dots = \frac{-\cos n\pi + 1}{n^2\pi} = \frac{(-1)^n + 1}{n^2\pi}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{2}{n^2\pi} & \text{if } n \text{ is even} \end{cases}$$

then repeat w/ b_n


Fourier Series
4
3.1
2.21
1.32
0.42
-0.47
-1.36
-2.25
-3.15
-3.141592653589793
-1.7802358370342148
-0.4188790204786382
0.9424777960769389
3.141592653589793
Independent variable: x
f(x)
Sf(x) with 3 terms

Quiz 7

1. Solve

$$\boldsymbol{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

Note that the solution to the corresponding homogeneous system is

$$\boldsymbol{x} = C_1 \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix} + C_2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

Hint: I wouldn't use undetermined coefficients

2. Use the Euler method $y_{n+1} = y_n + f(t_n, y_n)h$ to estimate $y(1)$ with a step size of $h = 0.5$

$$y' = 2y - 3t, \quad y(0) = 1$$