10.3 Fourier Convergence Theorem

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, 3, \ldots$$

does the Fourier series (FS) converge to $f(x)$?

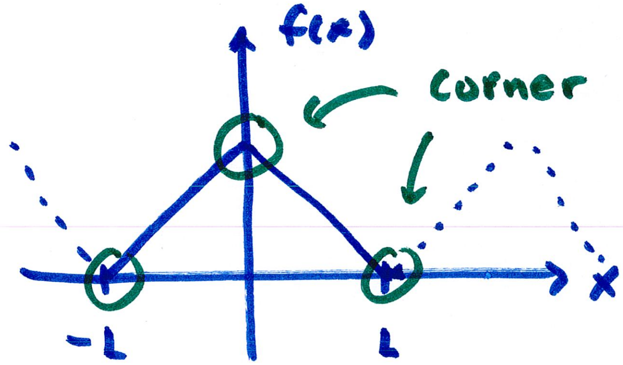
does FS converge to $f(x)$ at every x?

(answer is yes to both for Taylor series)

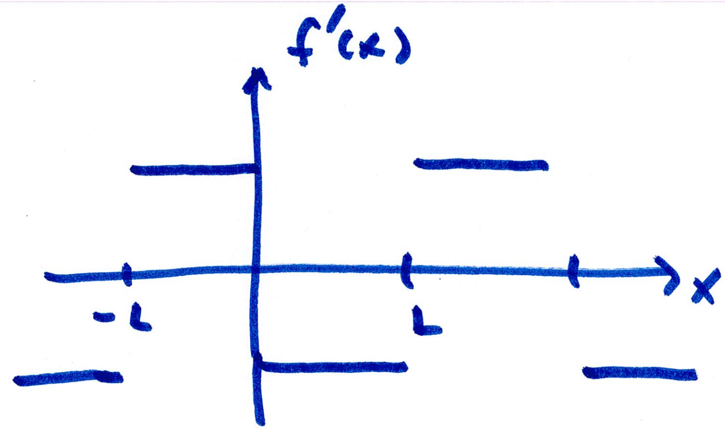
→ yes, if $f(x)$ is at least piecewise smooth

→ means $f'(x)$ can only have a finite number of discontinuities and/or corners/cusps/discontinuities

$f(x)$ has finite # of



this is piecewise smooth.



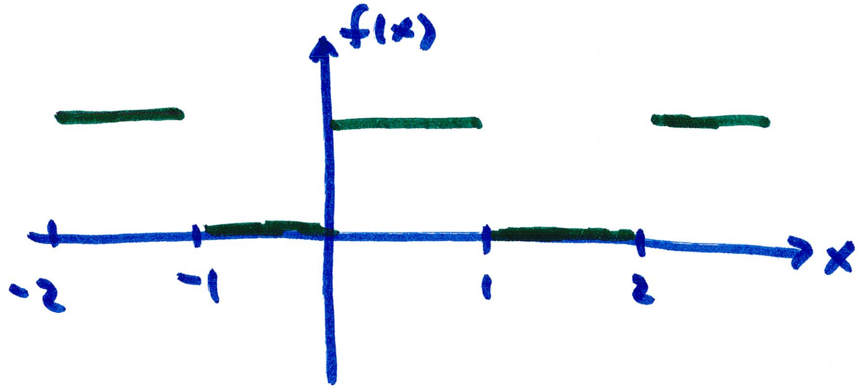
and $f(x)$ is assumed to be periodic w/ period of $2L$

(or has periodic extensions to make it periodic)

2nd question: does FS converge to $f(x)$ at every x?

example

$$f(x)=\begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \end{cases}$$



fundamental period = 2

$L = 1$

piecewise smooth, FS exists

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \int_0^1 1\,dx = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_0^1 \cos n\pi x\,dx$$

$$= \frac{1}{n\pi}\sin n\pi x\Big|_0^1 = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \int_0^1 \sin n\pi x\,dx$$

$$= -\frac{1}{n\pi}\cos n\pi x\Big|_0^1 = -\frac{1}{n\pi}(\cos n\pi - 1)$$

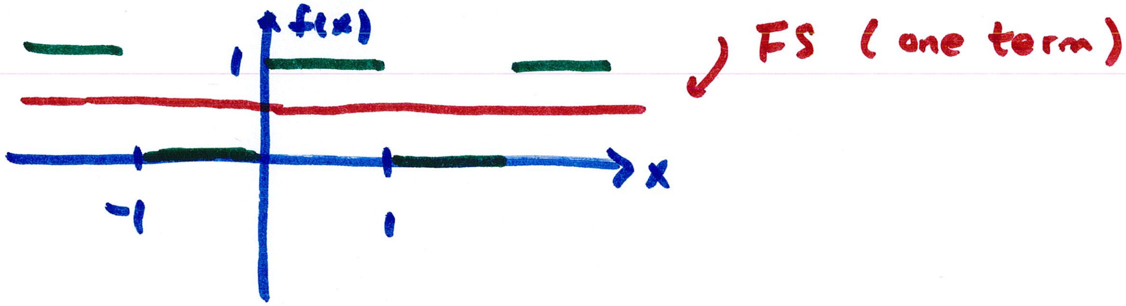
$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

let $n = 2k-1 \qquad k = 1, 2, 3, \ldots$

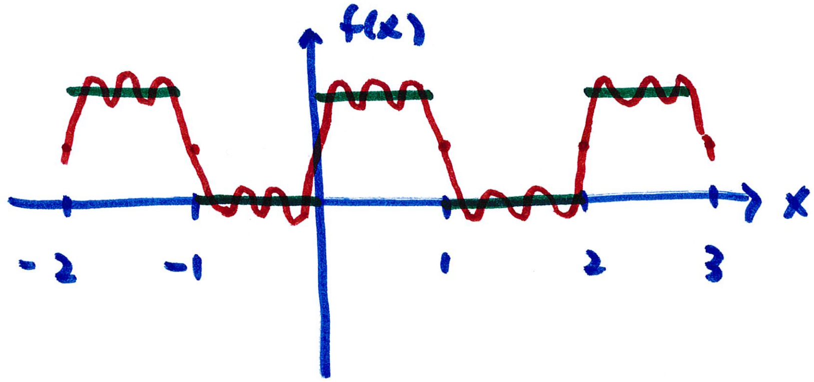
FS: $$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin\left[(2k-1)\pi x\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right)$$

one-term FS: $f(x) = \frac{1}{2}$



two-term FS: $f(x) = \frac{1}{2} + \frac{2}{\pi} \sin \pi x$

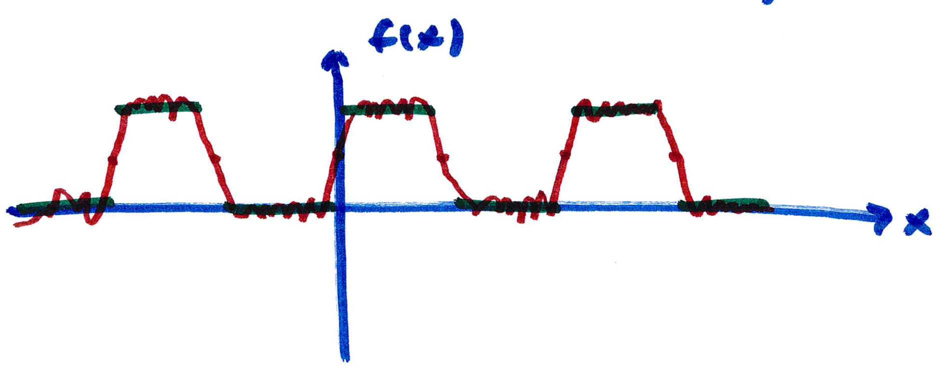


at discontinuities ($x = \pm 1, \pm 2, \pm 3, \ldots$)

$\sin \pi x = 0$

~~can~~ FS = f(x) at some x's (evenly-spaced wherever f(x) is continuous)

more terms



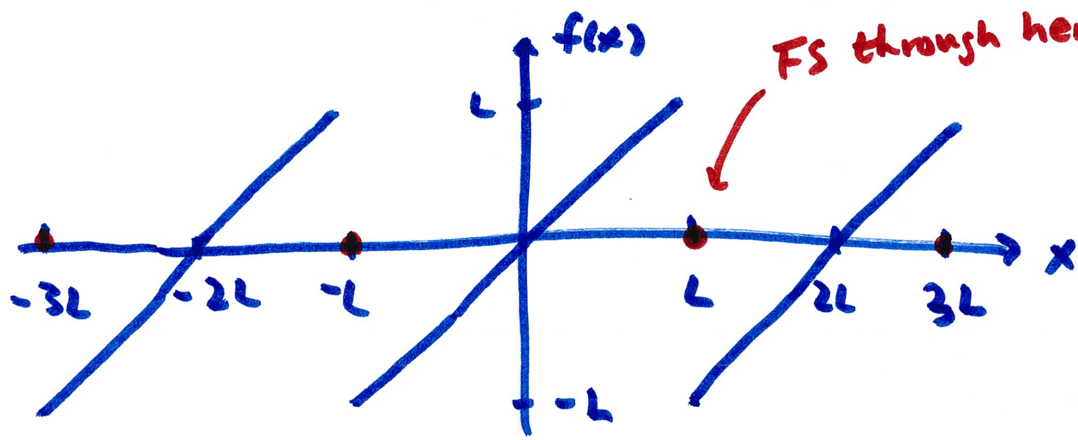
FS will converge to $f(x)$ at every x whenever $f(x)$ is <u>continuous</u>

FS will converge to the <u>average</u> $f(x)$ just to the right and left at discontinuities

→ the value of $\frac{1}{2}a_0$ ("mean value term")

<u>example</u>

$f(x) = x \qquad -L < x < L$



add periodic extensions

find a_0 by inspecting graph

FS through average of L and $-L$ → 0

$$\frac{1}{2}a_0 = 0 \rightarrow a_0 = 0$$

verify: $a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \frac{1}{L}\frac{x^2}{2}\Big|_{-L}^{L} = 0$

$a_n = \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}\,dx$ $\quad u = x \quad dv = \cos\frac{n\pi x}{L}dx$

$\quad du = dx \quad v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$

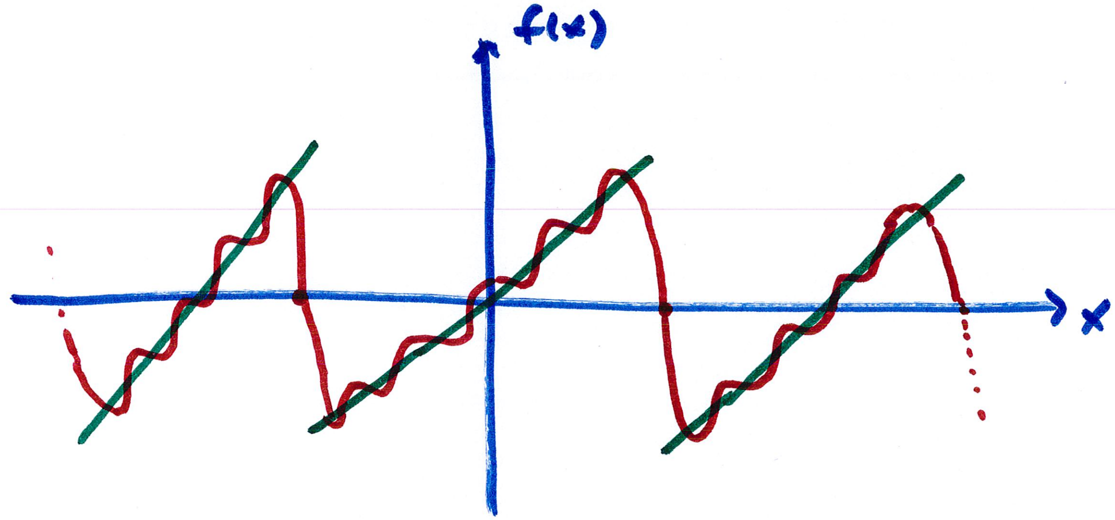
$= \frac{1}{L}\left(\cancel{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L}}^{\,0} - \frac{L}{n\pi}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx\right)$

$= -\frac{1}{n\pi}\cdot -\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} = 0$

$b_n = \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}\,dx$ $\quad$ by parts again

$= \cdots = -\frac{2L}{n\pi}\cos n\pi = -\frac{2L}{n\pi}(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}$

FS: $f(x) = \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$



more terms included will decrease ~~ther~~ the error

on $-L < x < L$

but the error at $x = \pm nL$ <u>cannot</u> be decreased

no matter how ~~big~~ many terms are in FS

("Gibb's phenomenon")