10.4 Even and Odd Functions

even function: $f(-x) = f(x)$

e.g. x^2, x^4, $\cos x$ y-axis symmetry

odd function: $f(-x) = -f(x)$

e.g. x^3, x^5, $\sin x$ origin symmetry

product / quotient of two even functions is even

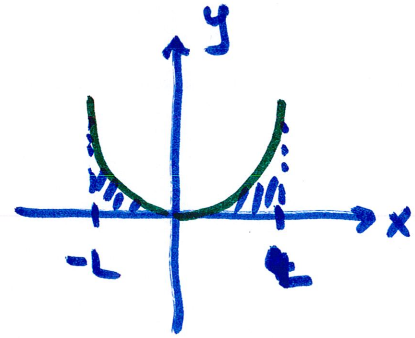
$$h(x) = f(x)\,g(x) \qquad h,\ f,\ g \text{ are even}$$

$$\begin{aligned} h(-x) &= f(-x)\,g(-x) \\ &= f(x)\,g(x) \\ &= h(x) \end{aligned}$$

product / quotient of two odd functions is even

product / quotient of an even and an odd is odd

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\, \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}\, dx$$

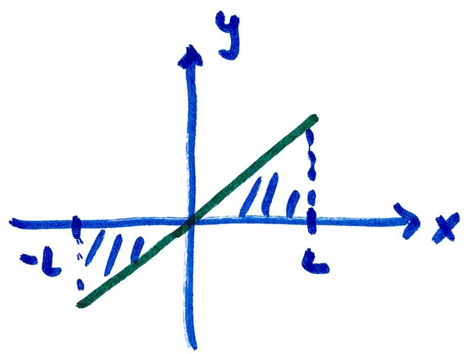


if $f(x)$ is even, then $f(x)\cos\frac{n\pi x}{L}$ is even

$$\int_{-L}^{L} (\text{even function})\, dx = 2\int_{0}^{L} (\text{even function})\, dx$$

if $f(x)$ is odd, then $f(x)\cos\frac{n\pi x}{L}$ is odd

$$\int_{-L}^{L} (\text{odd function})\, dx = 0$$



If $f(x)$ is <u>even</u>, then

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = 0$$

Fourier series is Fourier cosine series

If $f(x)$ is <u>odd</u>, then

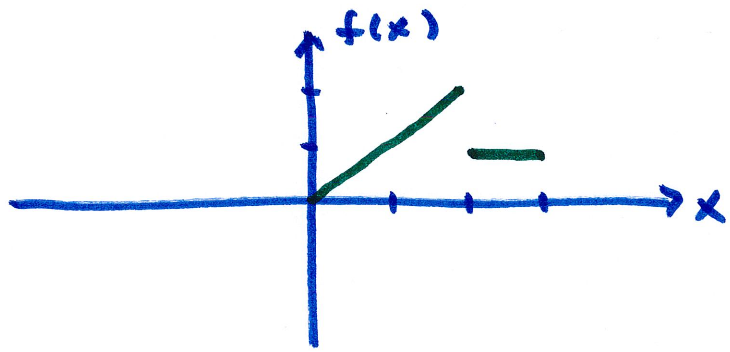
$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Fourier series is Fourier sine series

example

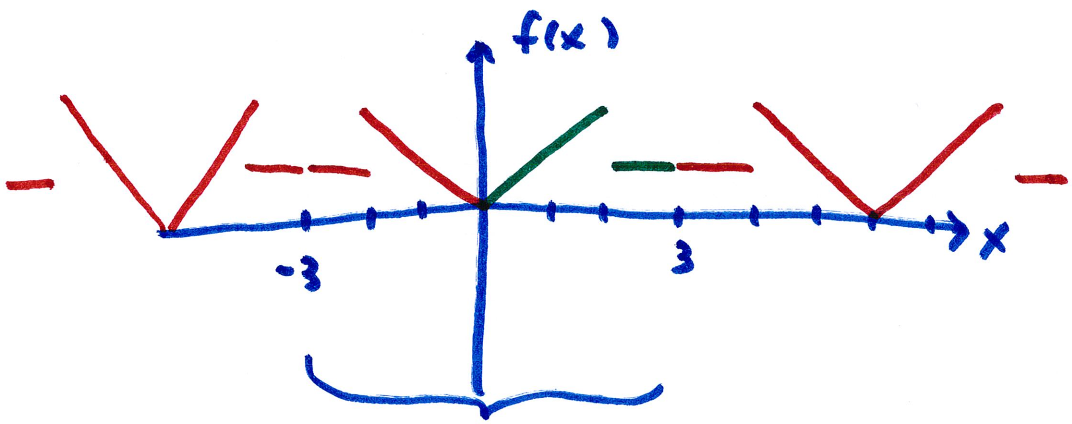
$$f(x) = \begin{cases} x & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 \le x < 3 \end{cases}$$



make this periodic then find Fourier series

add even or odd extension

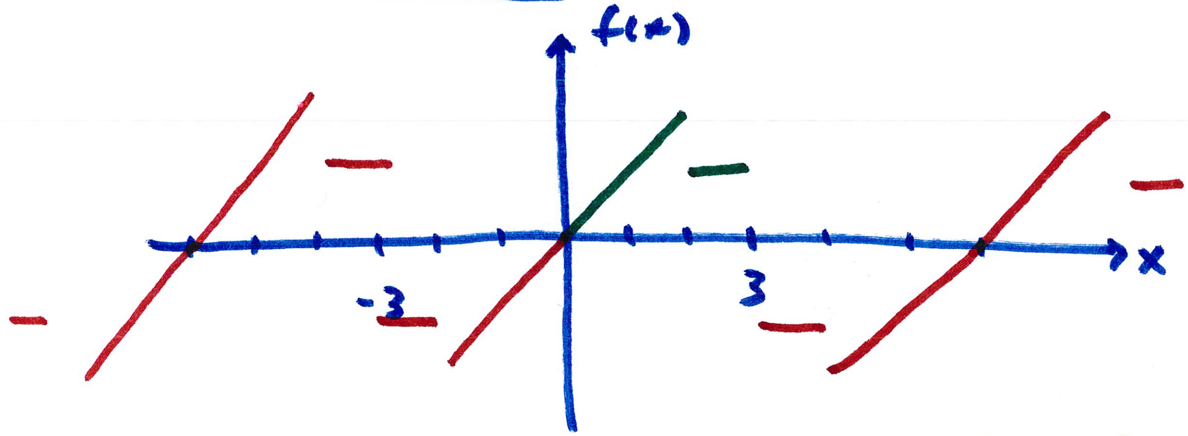
even extension



period = 6

period = 6

odd extension



period = 6

Fourier series for $f(x) = \begin{cases} x & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 \le x < 3 \end{cases}$ w/ even extension

period = 6 , $L = 3$

$b_n = 0$ $\quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$

$$a_n = \frac{2}{3} \left(\int_0^2 x \cos \frac{n\pi x}{3} \, dx + \int_2^3 1 \cdot \cos \frac{n\pi x}{3} \, dx \right)$$

$$a_0 = \frac{2}{3} \left(\int_0^2 x \, dx + \int_2^3 dx \right) = 2$$

by parts for first integral

$$a_n = \frac{2}{n\pi}\sin\left(\frac{2n\pi}{3}\right) + \frac{6}{n^2\pi^2}\left[\cos\left(\frac{2n\pi}{3}\right) - 1\right]$$

FS: $f(x) = 1 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$

FS of f(x) w/ odd extension

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

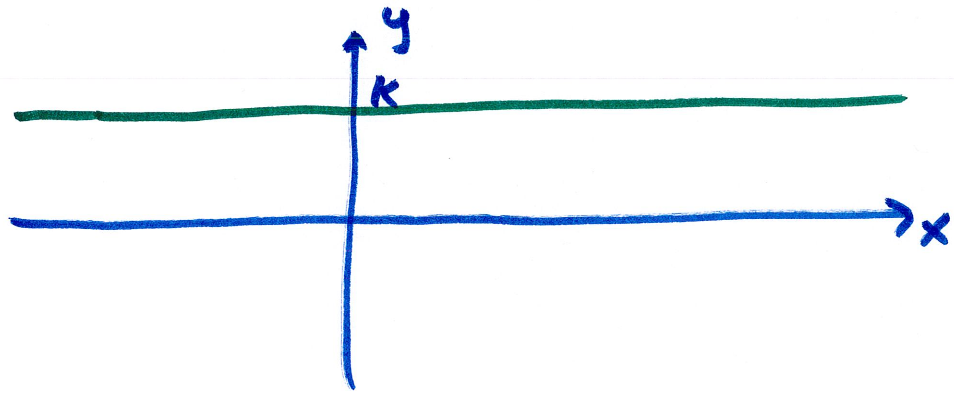
$$= \frac{2}{3}\left(\int_0^2 x\sin\frac{n\pi x}{3}\,dx + \int_2^3 \sin\frac{n\pi x}{3}\,dx\right)$$

$$= \cdots = \frac{6}{n^2\pi^2}\sin\left(\frac{2n\pi}{3}\right) - \frac{2}{n\pi}\left[\cos\left(\frac{2n\pi}{3}\right) + \cos(n\pi)\right]$$

FS: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$

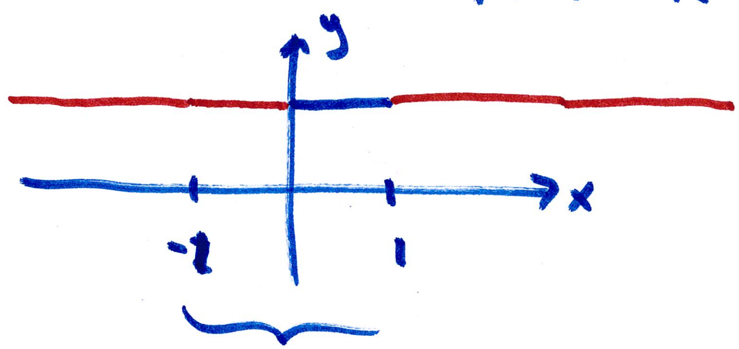
What is the Fourier series of a constant?

$f(x) = K$



define ~~$f(x) = \{ \; K$ if $0 \le x \le 1$~~

$f(x) = K \quad 0 \le x \le 1$ w/ <u>even extension</u>



period = 2 $\quad L = 1 \quad b_n = 0$

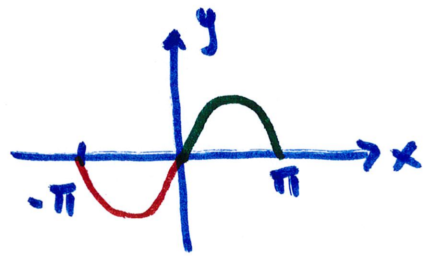
$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

$$a_0 = 2 \int_0^1 K \, dx = 2K \qquad a_n = 2 \int_0^1 K \cdot \cos n\pi x \, dx$$
$$= 0$$

How about Fourier series of $\sin x$?

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$\sin x = \sin x \quad 0 \le x \le \pi$ w/ odd extension



period $= 2\pi$

$L = \pi$

$a_n = 0$ (odd)

$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

only one $b_n \neq 0$

because $\sin nx$ is orthogonal to itself w/ different frequency

or: $\sin x = \frac{1}{2}\overset{=0}{a_0} + \overset{=0}{a_1} \cos\frac{1\pi x}{L} + \overset{=0}{a_2} \cos\frac{2\pi x}{L} + \dots$

$+ \underset{=1}{b_1} \sin\frac{\pi x}{L} + \underset{=0}{b_2} \sin\frac{2\pi x}{L} + \underbrace{\dots}_{=0} \qquad L = \pi$