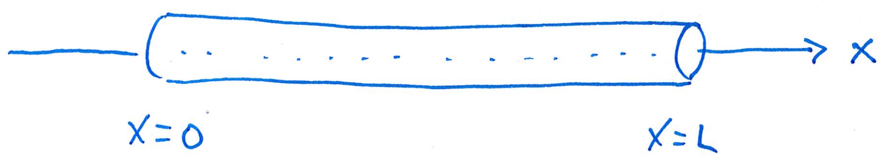
10.5 Separation of Variables and Heat Conduction in a Rod



thin metal rod so temperature is function of x and t only

$u(x,t)$

lateral surface is insulated, no lateral heat loss (no "y" component)

1-D Heat Equation

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad u_t = \frac{\partial u}{\partial t}$$

$$\alpha^2 u_{xx} = u_t$$

$$0 \le x \le L$$

partial differential eq (PDE)

α^2 : thermal diffusivity constant ($\alpha^2 = 1.14$ for aluminum)

Boundary conditions: $u(0,t) = T_1$ left end heated to T_1 for all t

$u(L,t) = T_2$

initial condition: $u(x,0) = f(x)$ initial temp profile in the rod

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Solve a simple case: $T_1 = T_2 = 0$

$$\alpha^2 u_{xx} = u_t$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

method of separation of variables

assume $u(x,t) = X(x)\,T(t)$

big X

$$u_x = \frac{\partial}{\partial x}(X T) = \frac{\partial X}{\partial x} T = X' T$$

$$u_{xx} = X'' T \qquad u_t = X T'$$

BC's: $u(0,t)=0 \rightarrow X(0)\,T(t)=0 \Rightarrow X(0)=0$

$u(L,t)=0 \rightarrow X(L)\,T(t)=0 \Rightarrow X(L)=0$

IC: $u(x,0)=f(x) \rightarrow \underbrace{X(x)\,T(0)=f(x)}_{\text{can't tell us about } T(0)}$ $\qquad ab=3$

Sub $u_{xx}=X''T$ and $u_t = XT'$ into $\alpha^2 u_{xx}=u_t$

$$\alpha^2 X''T = XT'$$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \qquad \text{for all } x, t$$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = \text{constant} \quad (\text{separation constant})$$

$$= -\lambda$$

$$\frac{X''}{X} = -\lambda \qquad \frac{T'}{\alpha^2 T} = -\lambda$$

solve $X'' + \lambda X = 0$ w/ BC's $X(0) = 0,\ X(L) = 0$

boundary value problem from 10.1

eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ $\quad n = 1, 2, 3, \ldots$

eigenfunctions $X_n = \sin\left(\frac{n\pi x}{L}\right)$

Solve $T' + \alpha^2 \lambda T = 0$

$$T' + \frac{\alpha^2 n^2 \pi^2}{L^2} T = 0$$ first order linear eq.

fundamental solution: $T_n = e^{-\alpha^2 n^2 \pi^2 t / L^2}$ $\quad n = 1, 2, 3, \ldots$

$u(x,t) = X T$

for each n, fundamental solution is

$$u_n = e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin\left(\frac{n\pi x}{L}\right)$$

general solution is linear combination of all u_n

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\alpha^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

find C_n using IC: $u(x,0) = f(x)$

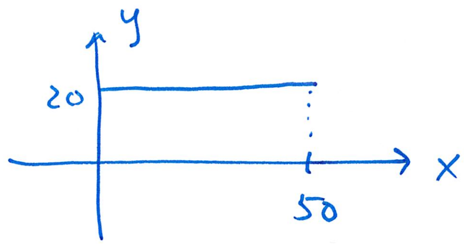
$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$ Fourier sine series

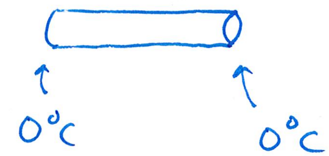
$$\boxed{C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

example $L = 50$ $\quad \alpha^2 = 1$ (aluminum/copper alloy)

both ends at 0

$f(x) = 20$ (initial temp. profile)





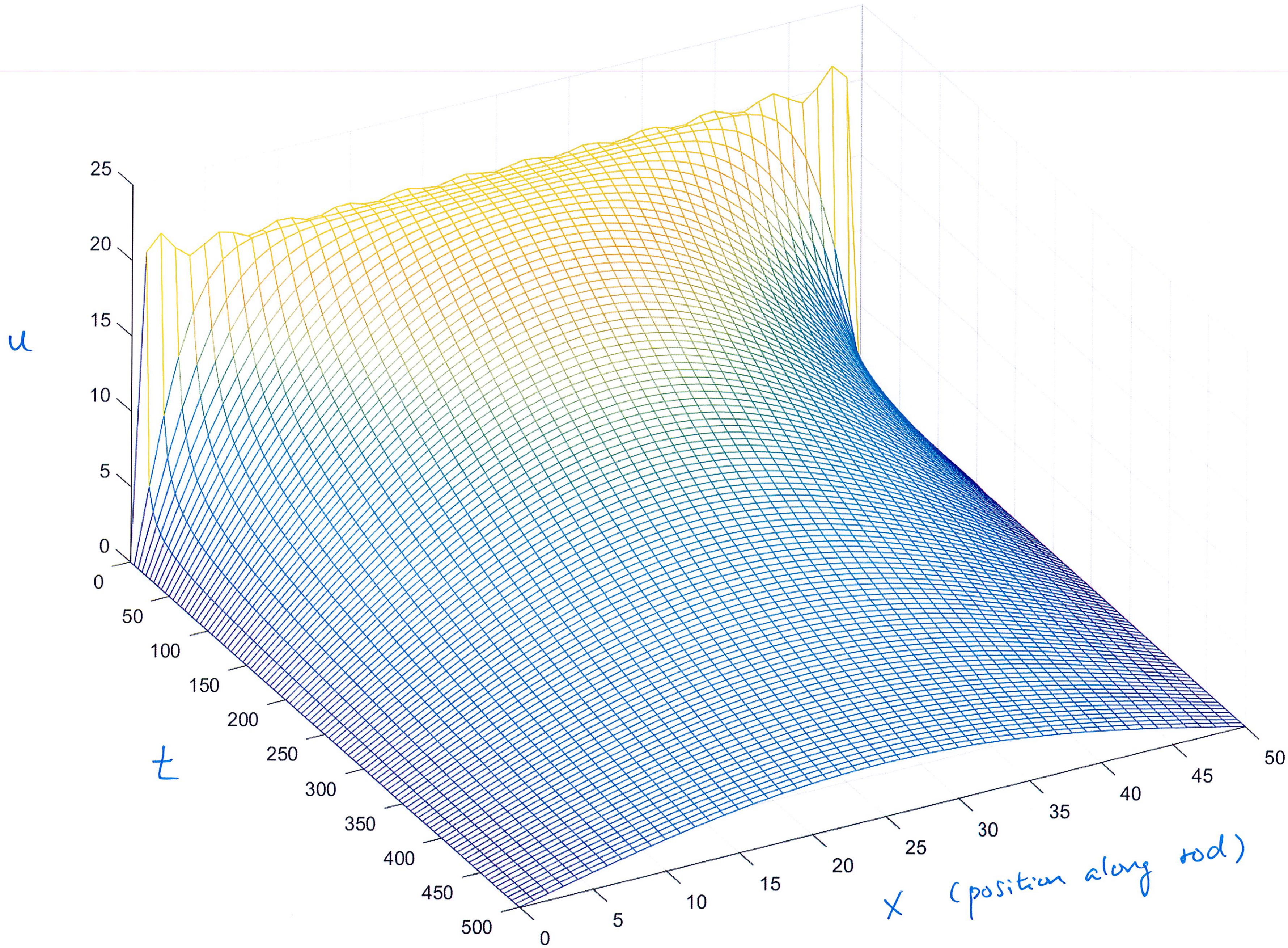
$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t/2500} \sin\left(\frac{n\pi x}{50}\right)$$

$$C_n = \frac{2}{50}\int_0^{50} 20 \cdot \sin\left(\frac{n\pi x}{50}\right) dx = \frac{40}{n\pi}(1-\cos n\pi)$$

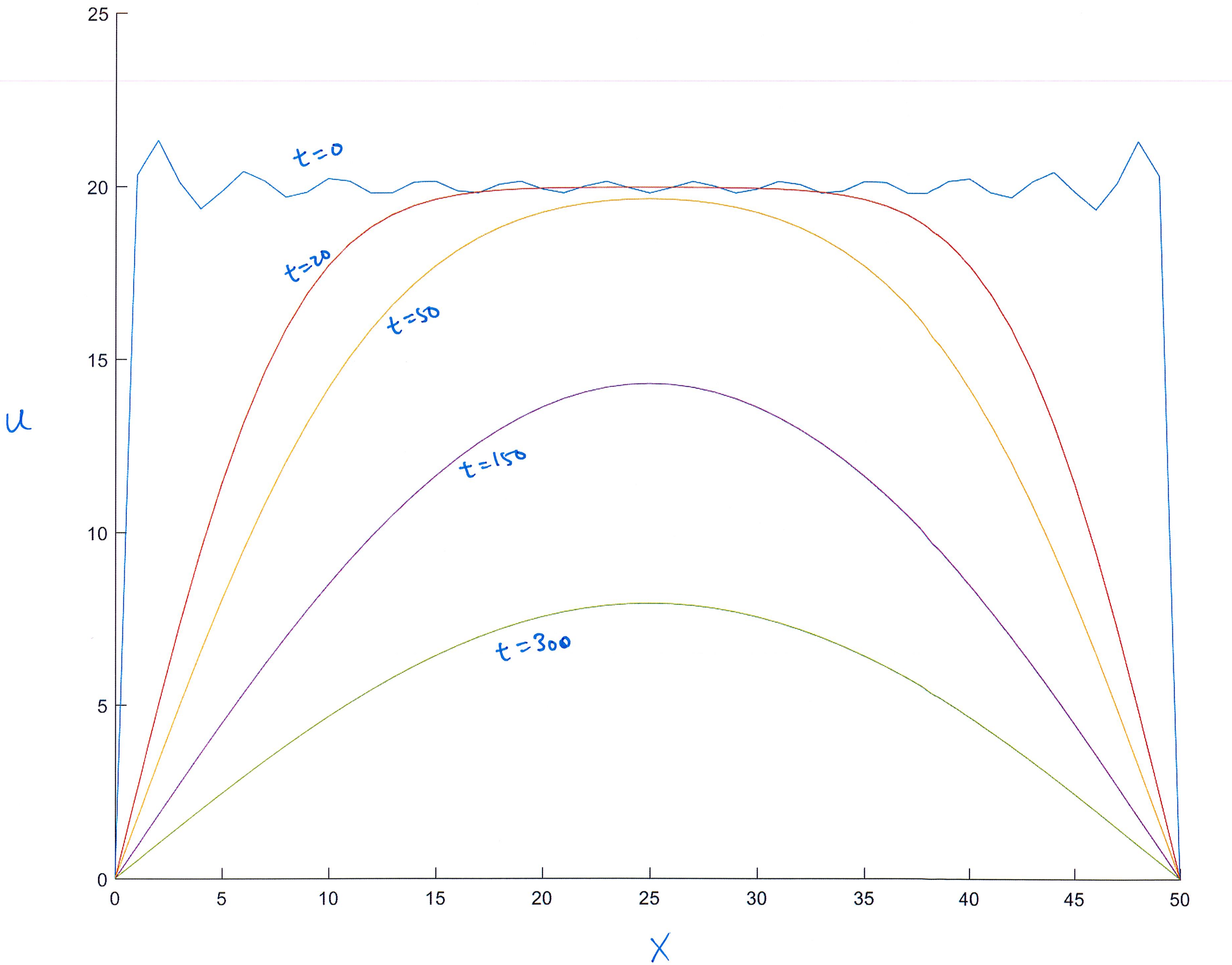
$n = 1, 2, 3, \ldots$

$$\lim_{t\to\infty} u(x,t) = 0$$

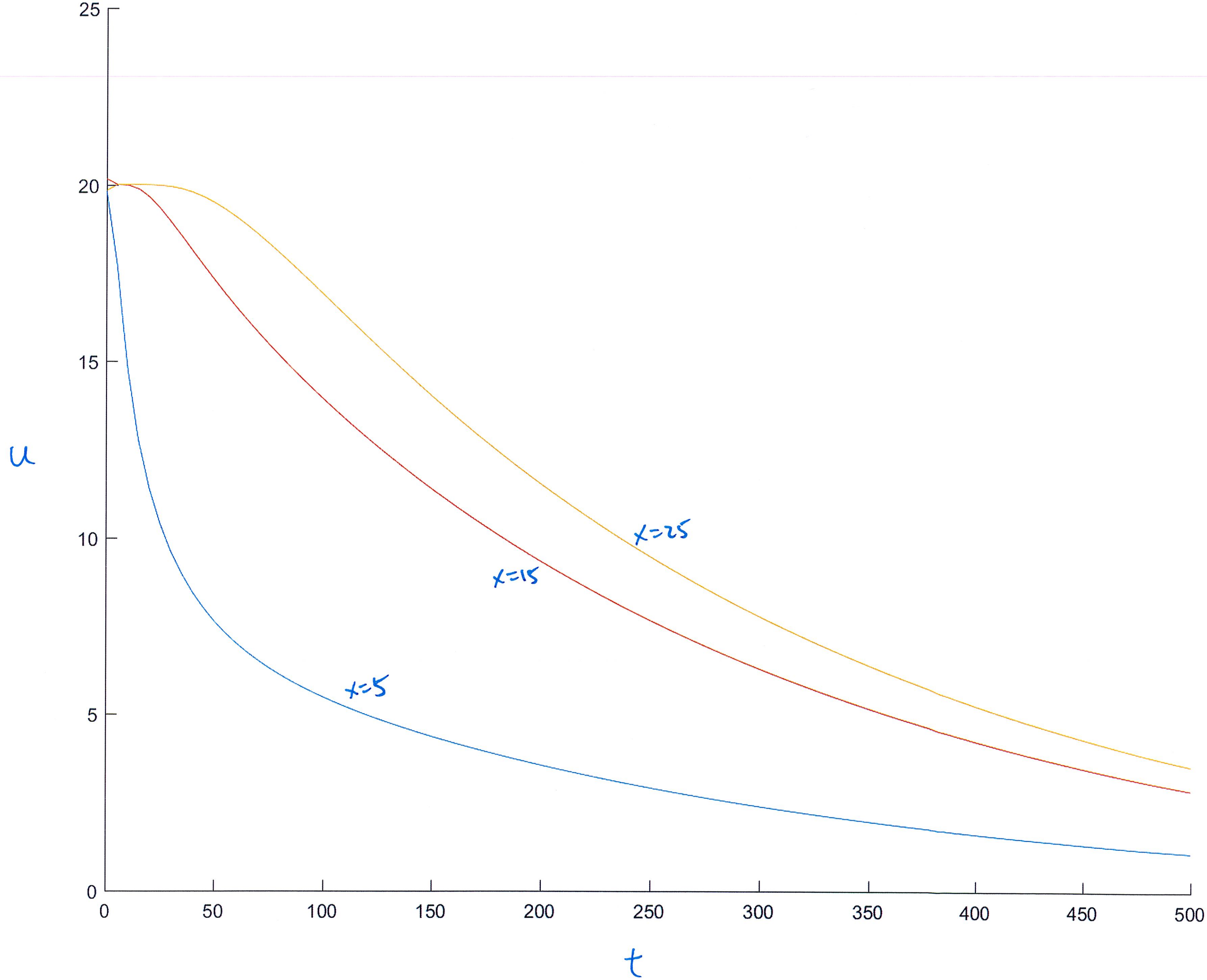
what does $u(x,t)$ look like before settling down to $0\,^\circ C$?



25
20
15
10
5
0
u
0
50
100
150
200
250
300
350
400
450
500
t
0
5
10
15
20
25
30
35
40
45
50
x (position along rod)



t=0
t=20
t=50
t=150
t=300
u
X
0
5
10
15
20
25
30
35
40
45
50



25
20
15
10
5
0
u
0
50
100
150
200
250
300
350
400
450
500
t
x=5
x=15
x=25

example

$L = 2 \qquad \alpha^2 = 4$

$$f(x) = 2\sin\left(\frac{\pi x}{2}\right) - \sin \pi x + 4 \sin 2\pi x$$

$$C_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

what is the FS of $f(x)$?

$$C_1 \sin\left(\frac{\pi x}{L}\right) + C_2 \sin\left(\frac{2\pi x}{L}\right) + C_3 \sin\left(\frac{3\pi x}{L}\right) + \dots$$

$L = 2$

$$C_1 \sin\left(\frac{\pi x}{2}\right) + C_2 \sin\left(\frac{2\pi x}{2}\right) + C_3 \sin\left(\frac{3\pi x}{2}\right) + \dots$$

$$C_1 = 2, \quad C_2 = -1, \quad C_3 = 0, \quad C_4 = 4, \quad C_5 = C_6 = C_7 = \dots 0$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \underbrace{e^{-\alpha^2 n^2 \pi^2 t / L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

negative exponential makes series converge quickly $\rightarrow$ often one or two terms are enough for most applications