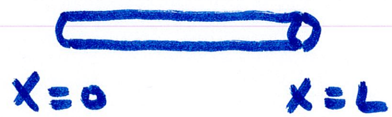
10.6 Other Heat Conduction Problems

$\alpha^2 u_{xx} = u_t$



last time: homogeneous BC's $\quad u(0,t) = u(L,t) = 0$

initial condition $\quad u(x,0) = f(x)$

solution: $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

now consider nonhomogeneous BC's: $\quad u(0,t) = T_1$

$u(L,t) = T_2$

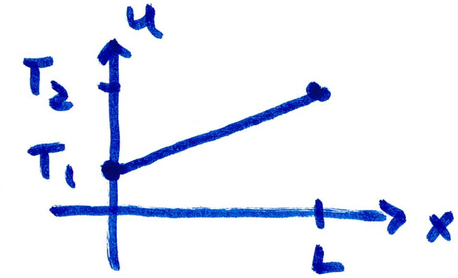
$$\alpha^2 u_{xx} = u_t \qquad 0 \le x \le L$$

$$u(0,t) = T_1 \qquad u(L,t) = T_2$$

$$u(x,0) = f(x)$$

use a change of variables to transform into homogeneous problem (solution is known)

let $v(x) = T_1 + \dfrac{T_2 - T_1}{L} x$



$$u(x,t) - v(x) = w(x,t)$$

BC's for $w(x,t)$: $\quad w(0,t) = 0 \qquad w(L,t) = 0$

so $w(x,t)$ has homogeneous BC's

IC: $\quad w(x,0) = u(x,0) - v(x) = f(x) - v(x)$

$$\left.\begin{aligned} u_{xx} &= w_{xx} \\ u_t &= w_t \end{aligned}\right\} \quad \alpha^2 w_{xx} = w_t$$

reuse solution from last time for w

$$w(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$u(x,t) = V(x) + w(x,t)$$

$$\boxed{u(x,t) = \left(T_1 + \frac{T_2 - T_1}{L} x\right) + \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

IC: $\cancel{u(x,0) = f(x)}$

$$w(x,0) = u(x,0) - V(x) = f(x) - V(x)$$

$$\left[f(x) - V(x)\right] = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{Fourier sine series}$$

$$C_n = \frac{2}{L}\int_0^L \left[f(x) - V(x)\right] \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\boxed{C_n = \frac{2}{L}\int_0^L \left[f(x) - \left(T_1 + \frac{T_2 - T_1}{L} x\right)\right] \sin\left(\frac{n\pi x}{L}\right) dx}$$

note: if $T_1 = T_2 = 0$, same solution as last time example $L=20$ $\alpha^2 = 0.86$ (aluminum)

$u(0,t)=0$ left at $0°c$

$u(20,t)=60$ right at $60°c$

$u(x,0)=25$ entire rod heated to $25°c$ at $t=0$

$$u(x,t) = 3x + \sum_{n=1}^{\infty} C_n e^{-0.86 n^2 \pi^2 t/400} \sin\left(\frac{n\pi x}{20}\right)$$

$$C_n = \frac{2}{20}\int_0^{20} (25-3x)\sin\frac{n\pi x}{20}\,dx$$ by parts

$$= \frac{70\cos n\pi + 50}{n\pi} \quad n = 1, 2, 3, \ldots$$

as $t \to \infty$, $u(x,t) = 3x = T_1 + \frac{T_2 - T_1}{L}x$

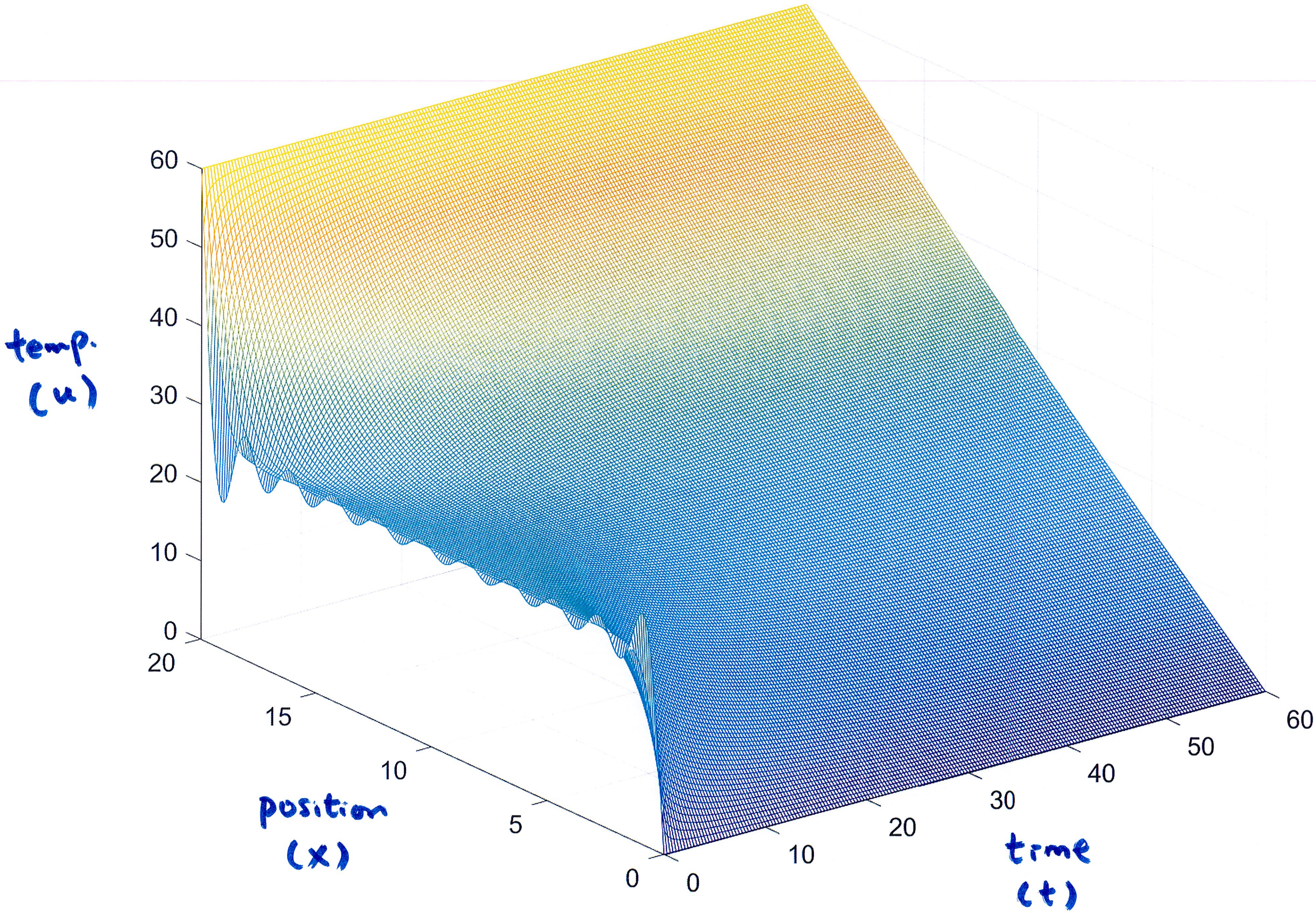
this is the steady state solution

(t is not relevant any more)

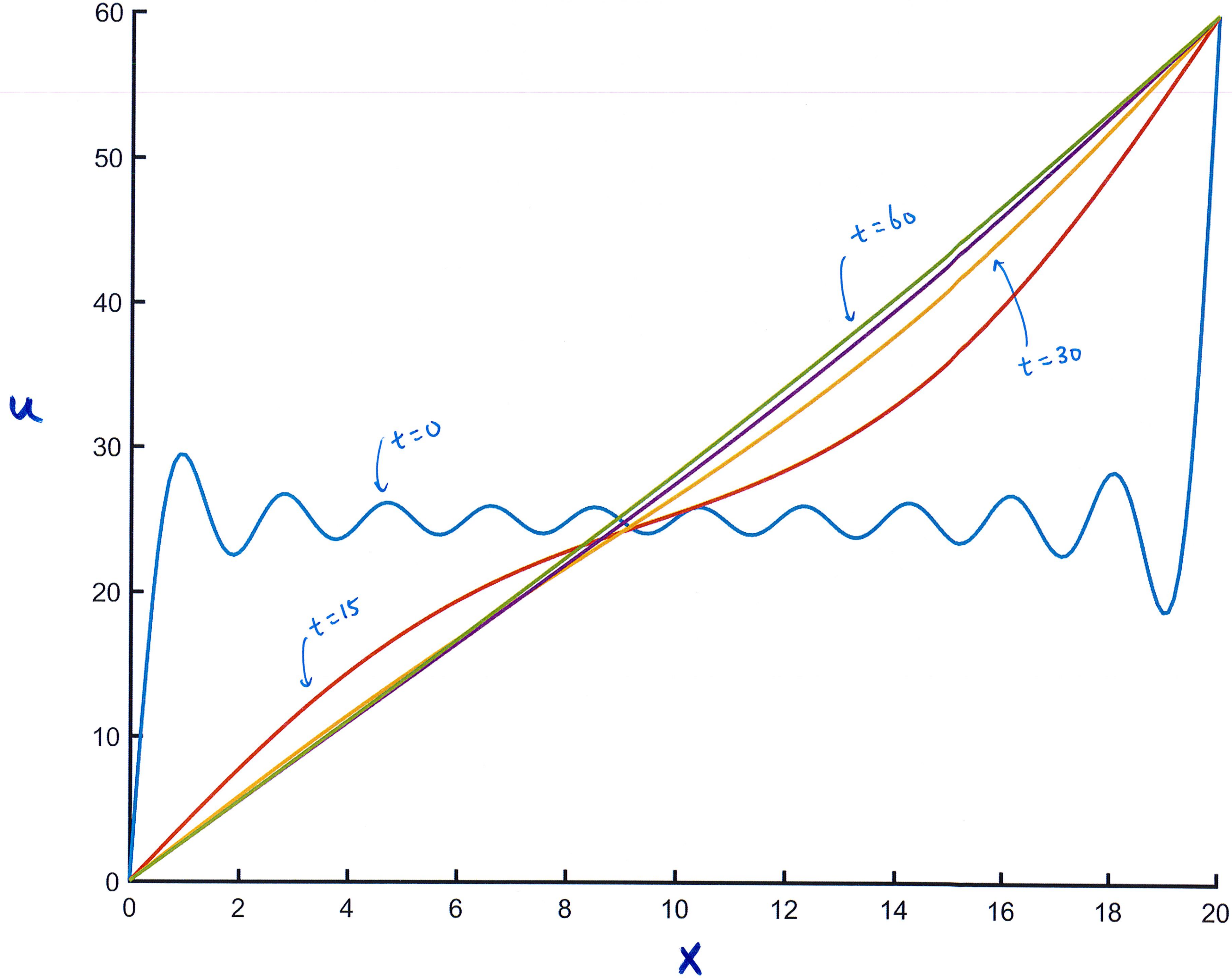
$u_t = 0$

$\alpha^2 u_{xx} = u_t = 0 \rightarrow u_{xx} = 0 \rightarrow u = C_1 + C_2 x$

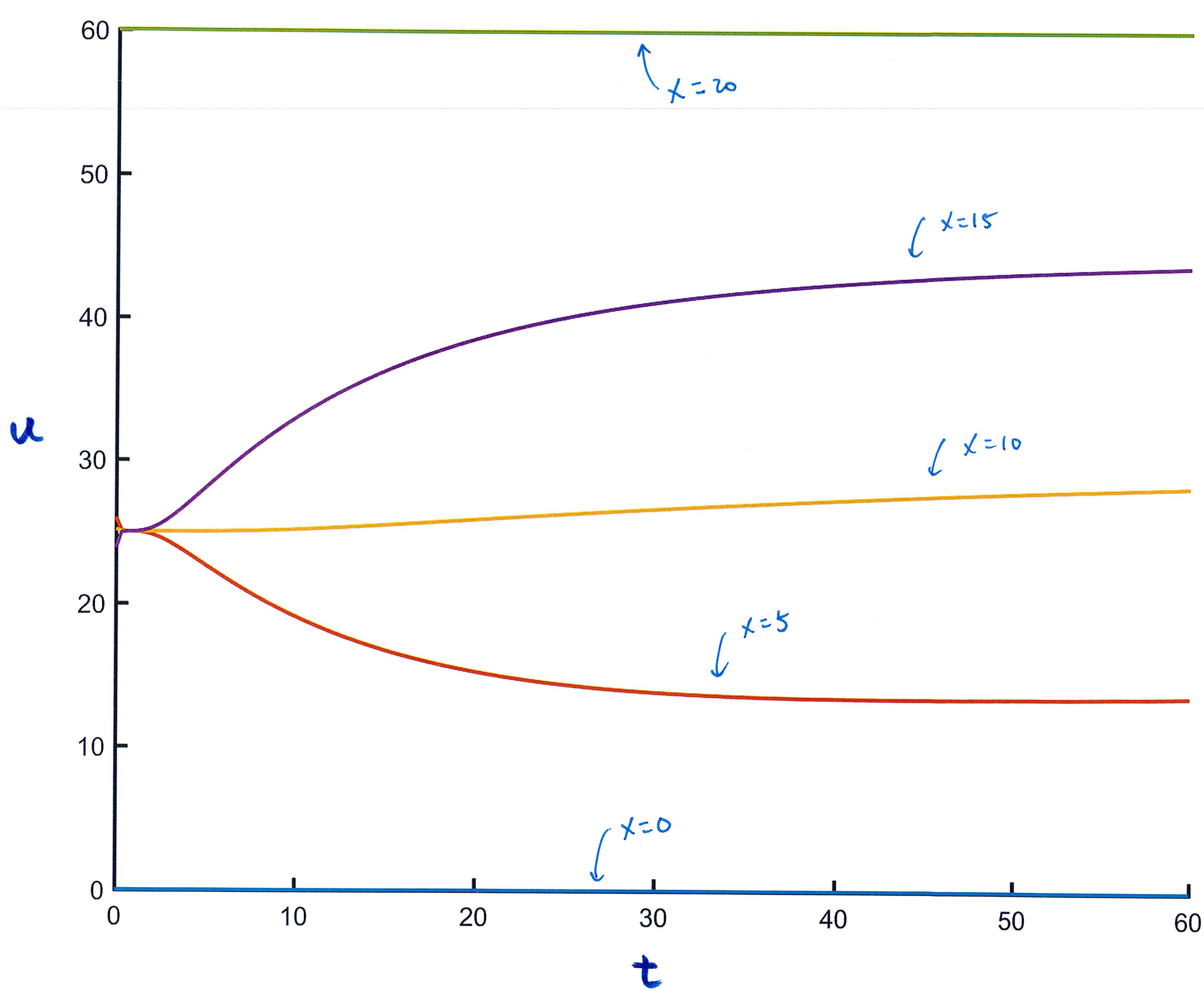
when t still matters, transient solution



temp. (u)
60
50
40
30
20
10
0
20
15
10
5
0
position (x)
0
10
20
30
40
50
60
time (t)

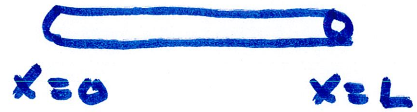


u
x
t=0
t=15
t=30
t=60
60
50
40
30
20
10
0
0
2
4
6
8
10
12
14
16
18
20



60
50
40
30
20
10
0
0
10
20
30
40
50
60
u
t
x=20
x=15
x=10
x=5
x=0

Insulated ends



heat can no longer escape through ends

$u_x(0,t) = u_x(L,t) = 0$

$\alpha^2 u_{xx} = u_t$

BC's: $u_x(0,t) = u_x(L,t) = 0$

IC: $u(x,0) = f(x)$

we can only specify 3 conditions

can't specify T_1, T_2

$u(x,t) = X(x)\,T(t)$

$u_{xx} = X''T \qquad u_t = XT'$

$\alpha^2 X''T = XT'$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -\lambda$$

$X'' + \lambda X = 0$

$T' + \alpha^2 \lambda T = 0$

solve $X'' + \lambda X = 0$

$u_x(0,t) = 0 \rightarrow X'(0)T(t) = 0$

$u_x(L,t) = 0 \rightarrow X'(L)T(t) = 0$

so $X'(0) = 0$, $X'(L) = 0$

$X(x) = k_1 \cos\sqrt{\lambda}\,x + k_2 \sin\sqrt{\lambda}\,x$

$X'(x) = -k_1\sqrt{\lambda}\sin\sqrt{\lambda}\,x + k_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$

$0 = k_2\sqrt{\lambda} \rightarrow k_2 = 0$

$0 = k_1\sqrt{\lambda}\sin\sqrt{\lambda}\,L = 0 \rightarrow \sin\sqrt{\lambda}\,L = 0$

$\lambda = 0$ is ok

$\sqrt{\lambda}\,L = n\pi \quad n = 0, 1, 2, 3, \ldots$

$\lambda = \left(\frac{n\pi}{L}\right)^2$ eigenvalues

$X_n = \cos\frac{n\pi x}{L}$ eigenfunctions

$$T' + \alpha^2 \lambda T = 0$$

$$T' + \frac{\alpha^2 n^2 \pi^2}{L^2} T = 0$$

$$T_n = e^{-\alpha^2 n^2 \pi^2 t / L^2} \qquad n = 0, 1, 2, 3, \ldots$$

fundamental solution for u:

$$u_n = e^{-\alpha^2 n^2 \pi^2 t / L^2} \cos\left(\frac{n\pi x}{L}\right) \qquad n = 0, 1, 2, 3,$$

$$u_0 = 1 \qquad \text{(eigenfunction)}$$

general solution:

$$\boxed{u(x,t) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2 \alpha^2 \pi^2 t / L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

Fourier cosine series

IC: $u(x,0) = f(x)$

$$f(x) = \frac{1}{2} c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\boxed{c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}$$

steady state: $t \to \infty$

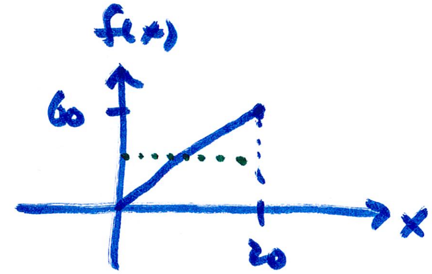
$$u(x,t) = \frac{1}{2} c_0$$

$$c_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

initial temp. profile

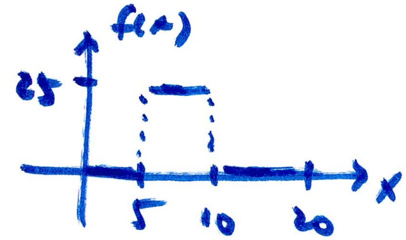
$$\text{Steady state temp: } \frac{1}{2} c_0 = \underbrace{\frac{1}{L} \int_0^L f(x)\, dx}_{\text{average of } f(x) \text{ on } 0 \le x \le L}$$

if $f(x) = 3x$ $L = 20$



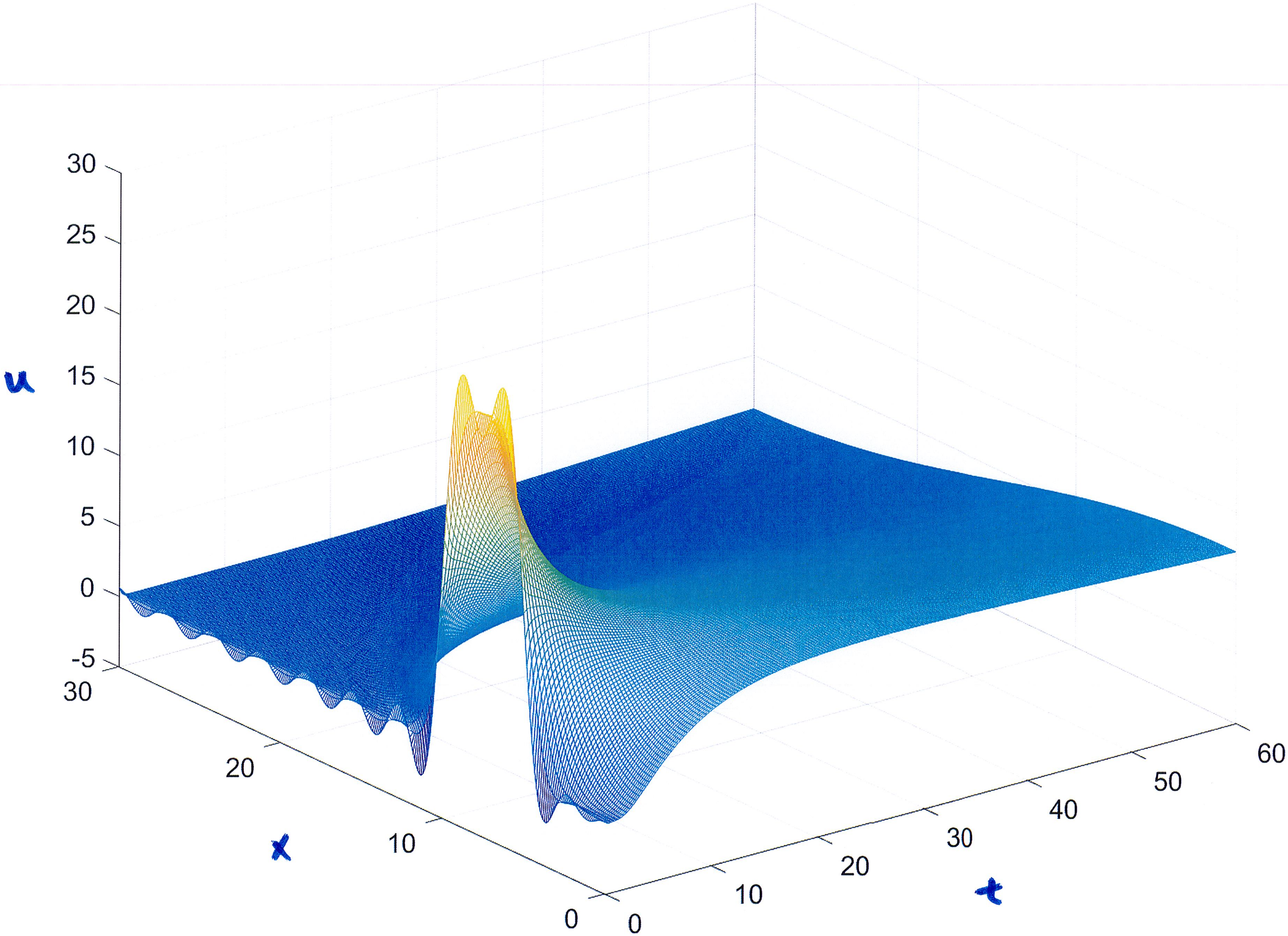
example $L = 30$ $\alpha^2 = 1$ ends insulated

$$f(x) = \begin{cases} 0 & 0 \le x < 5 \\ 25 & 5 \le x < 10 \\ 0 & 10 \le x \le 30 \end{cases}$$

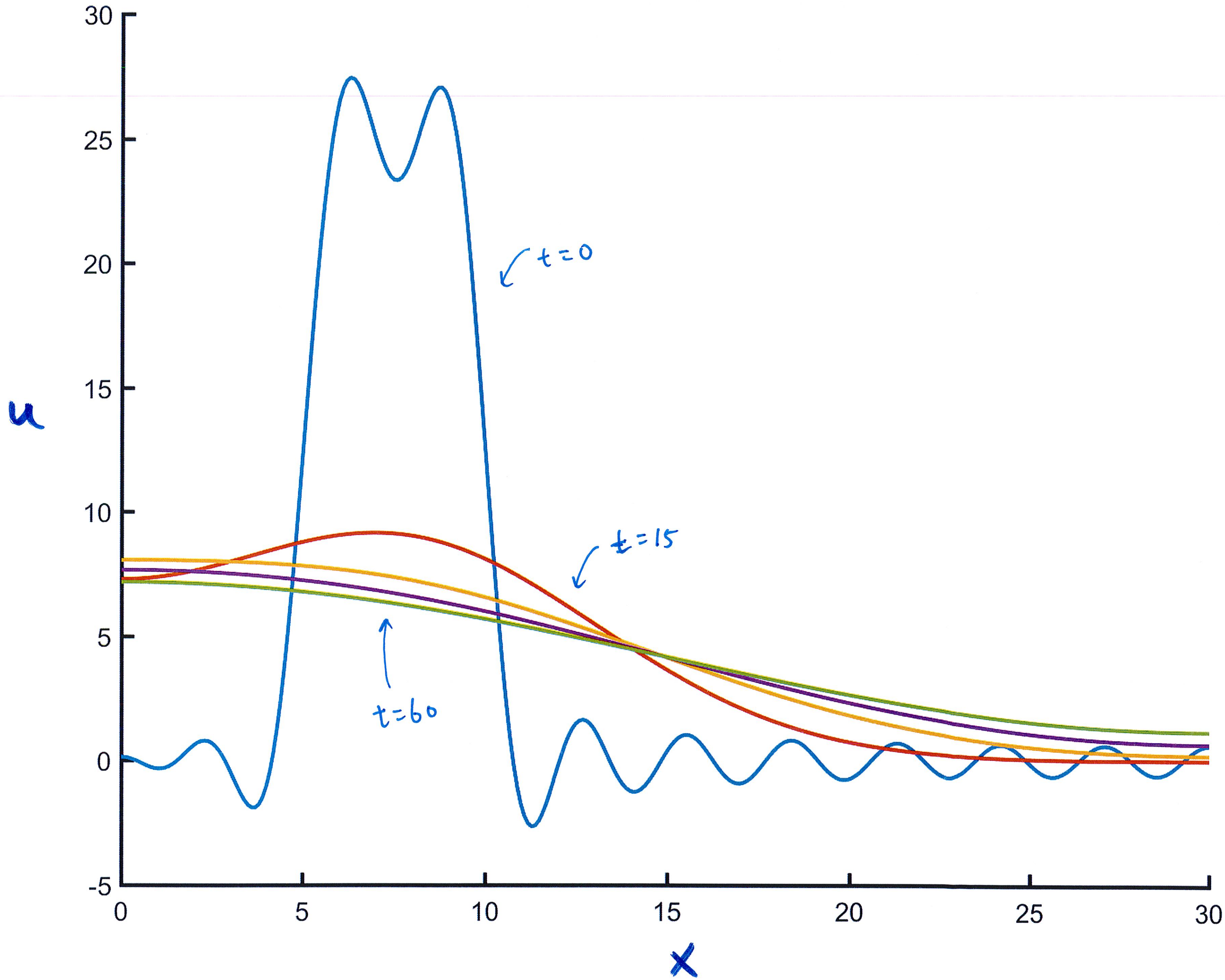


$$u(x,t) = \frac{25}{6} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/900} \cos\left(\frac{n\pi x}{30}\right)$$

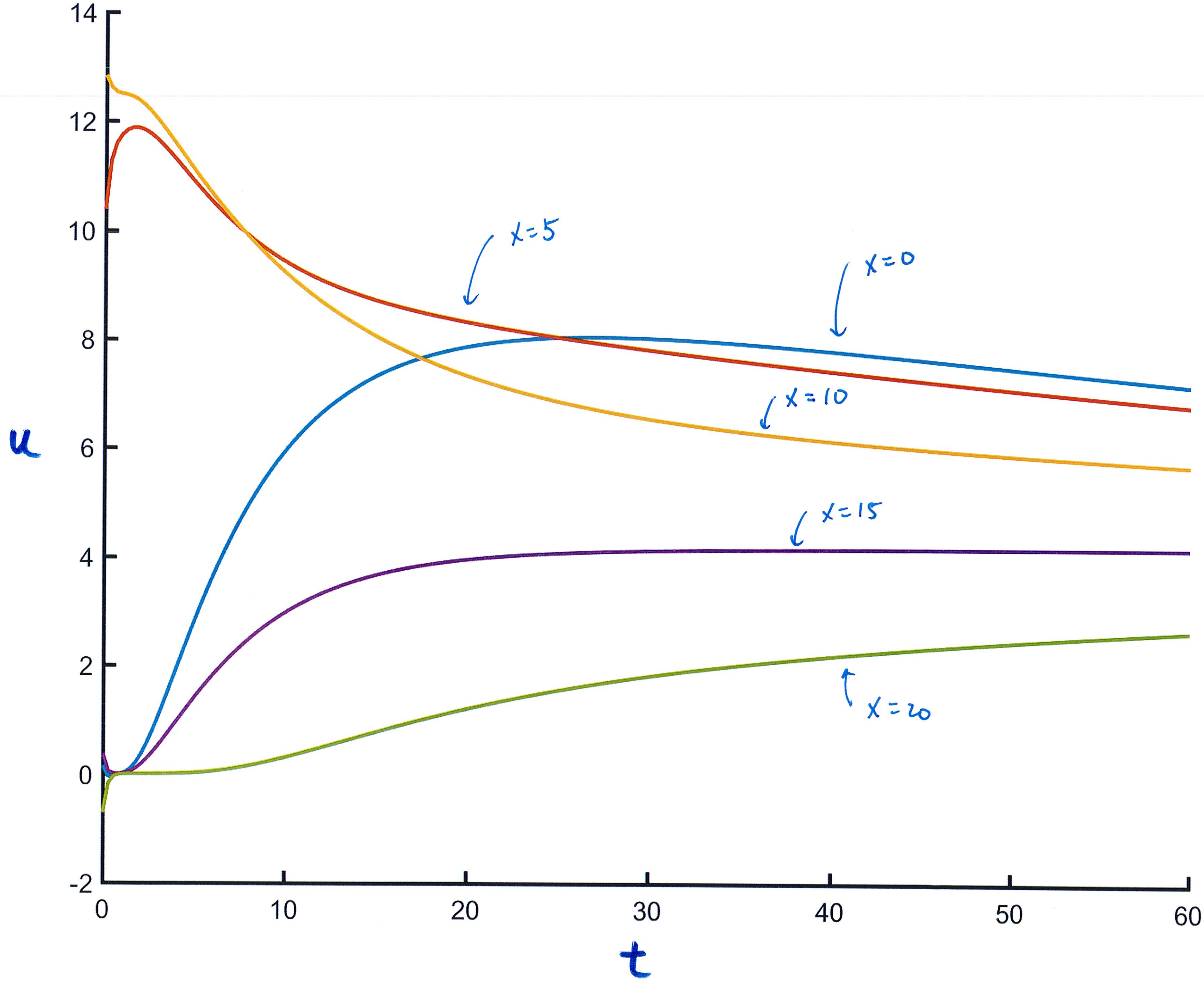
$$c_n = \frac{50}{n\pi} \left(\sin \frac{n\pi}{6} \right)$$



u
30
25
20
15
10
5
0
-5
x
30
20
10
0
t
0
10
20
30
40
50
60



30
25
20
15
10
5
0
-5
0
5
10
15
20
25
30
u
x
t=0
t=15
t=60


14
12
10
8
6
4
2
0
-2
0
10
20
30
40
50
60
u
t
x=5
x=0
x=10
x=15
x=20