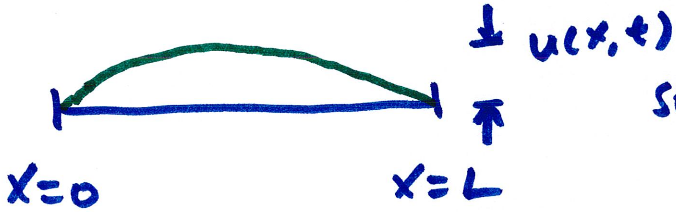
10.7 The Wave Equation: Vibrations in an Elastic String

HW for 10.7 is due Thu. 7/27



String L cm long

$u(x,t)$: displacement from equilibrium

equation: $a^2 u_{xx} = u_{tt}$

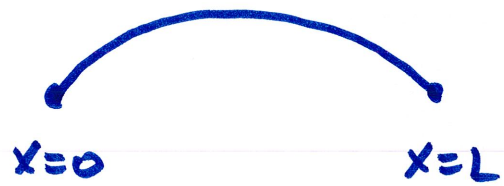
a^2 has unit of velocity

derivation: Newton's 2nd Law on infinitesimal segments.

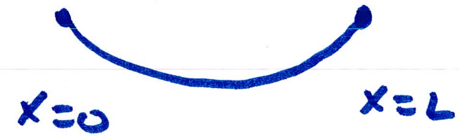
related to $\dfrac{\text{tension}}{\text{density}}$

physical meaning of $a^2 u_{xx} = u_{tt}$

u_{xx} is the concavity of shape of displaced string



$u_{xx} < 0$



$u_{xx} > 0$

u_{tt} is the acceleration (force) from the internal tension of string. $u_{tt} > 0$ is <u>upward</u> force

$a^2 u_{xx} = u_{tt}$ means u_{xx} and u_{tt} have same sign

$u_{xx} > 0$ means $u_{tt} > 0$

downward displacement means upward restoration force

at equilibrium, $u_{xx} = 0$, $u_{tt} = 0$ (no force)

$a^2 u_{xx} = u_{tt}$

separation of variables technique

(2 boundary value problem eqs. $\rightarrow$ $\underline{\underline{4}}$ conditions)

BC's: $u(0,t) = 0$ \
$u(L,t) = 0$ $\Big\}$ ends don't move

IC's: $u(x,0) = f(x)$ initial displacement

$u_t(x,0) = g(x)$ initial velocity

<u>Case 1: $f(x) \neq 0$, $g(x) = 0$</u> (no initial velocity)

$u(x,t) = X(x)\,T(t)$ $\quad u_{xx} = X''T \quad u_{tt} = XT''$

$a^2 X''T = XT''$

separate: $\dfrac{X''}{X} = \dfrac{T''}{a^2 T} = -\lambda$

$$X'' + \lambda X = 0 \qquad u(0,t)=0 \to X(0)T(t)=0 \to X(0)=0$$

$$u(L,t)=0 \to X(L)T(t)=0 \to X(L)=0$$

same as in heat eq.

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \ldots$$

$$X_n = \sin\frac{n\pi x}{L}$$

$$T'' + a^2\lambda T = 0$$

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0 \qquad u(x,0) = f(x)$$

$$u_t(x,0) = 0$$

$$X(x)T'(0) = 0 \to T'(0) = 0$$

$$T(t) = k_1 \cos\frac{n\pi a t}{L} + k_2 \sin\frac{n\pi a t}{L}$$

$$T'(t) = -k_1 \frac{n\pi a}{L} \sin\frac{n\pi a t}{L} + k_2 \frac{n\pi a}{L} \cos\frac{n\pi a t}{L}$$

$$0 = k_2 \frac{n\pi a}{L} \longrightarrow k_2 = 0$$

$$T_n = \cos\frac{n\pi a t}{L}$$

fundamental solution for $u(x,t)$:

$$u_n(x,t) = \cos\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}$$

n^{th} mode or
n^{th} harmonic

$n=1$: fundamental mode
first harmonic

$$\boxed{\text{gen. solution: } u(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}}$$

$\frac{n\pi a}{L}$: frequency of n^{th} mode

$\frac{2L}{n}$: wavelength of n^{th} mode

IC: $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

sine series

$$\boxed{C_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

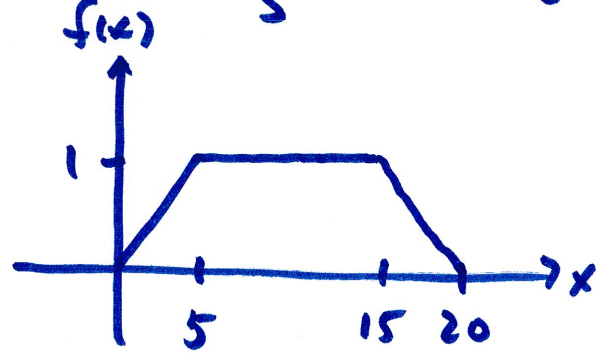
example $L=20 \quad a^2=1$

$u(0,t)=u(20,t)=0$

$u(x,0)=f(x)$

$u_t(x,0)=0$

$$f(x)=\begin{cases} \frac{1}{5}x & 0\le x<5 \\ 1 & 5\le x<15 \\ \frac{20-x}{5} & 15\le x\le 20 \end{cases}$$



$$u(x,t)=\sum_{n=1}^{\infty} C_n \cos\frac{n\pi}{20}t \sin\frac{n\pi x}{20}$$

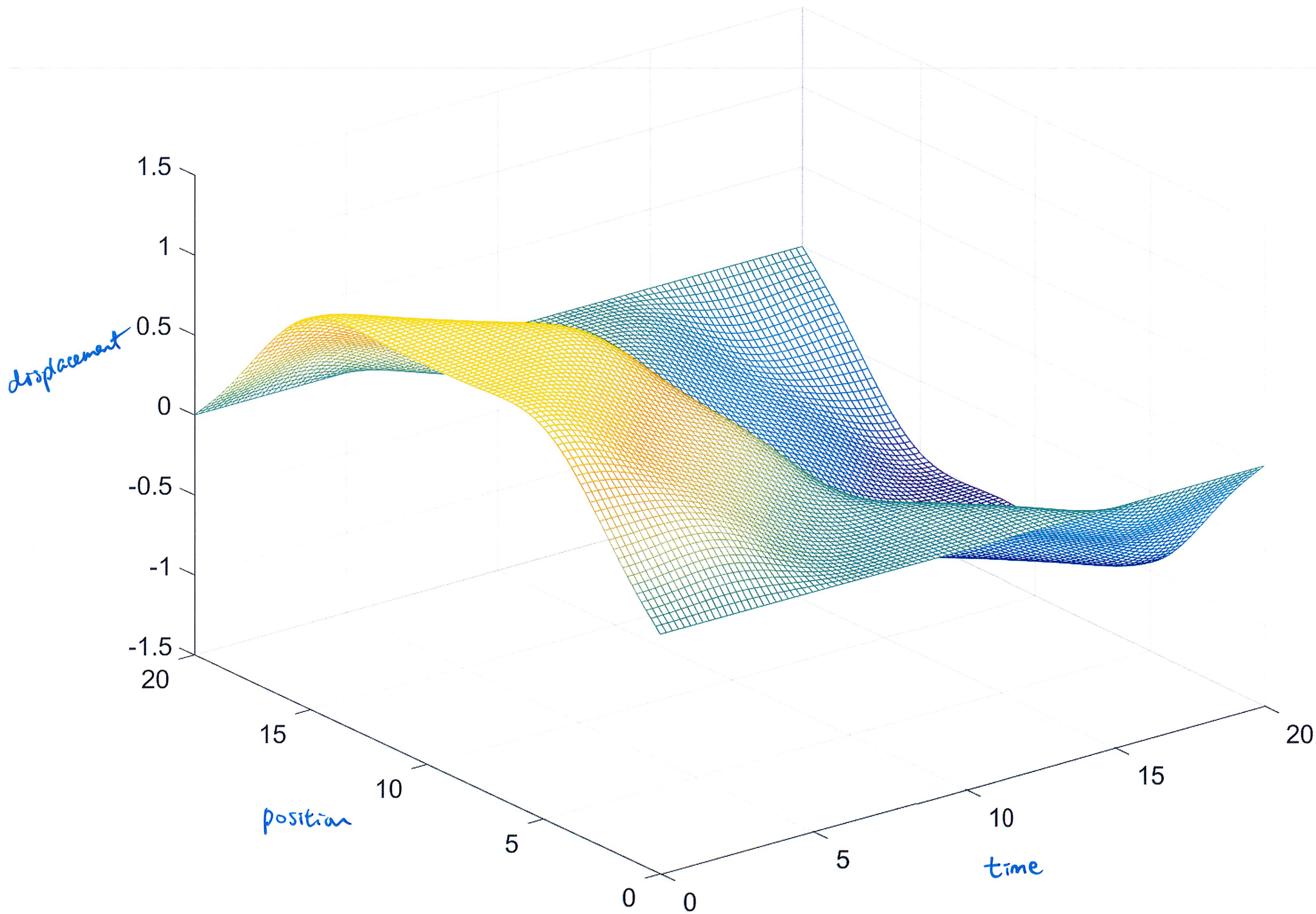
$$C_n=\frac{2}{20}\int_0^{20} f(x)\sin\frac{n\pi x}{20}\,dx$$

$$=\cdots=\frac{8}{n^2\pi^2}\left(\sin\frac{n\pi}{4}+\sin\frac{3n\pi}{4}\right)$$

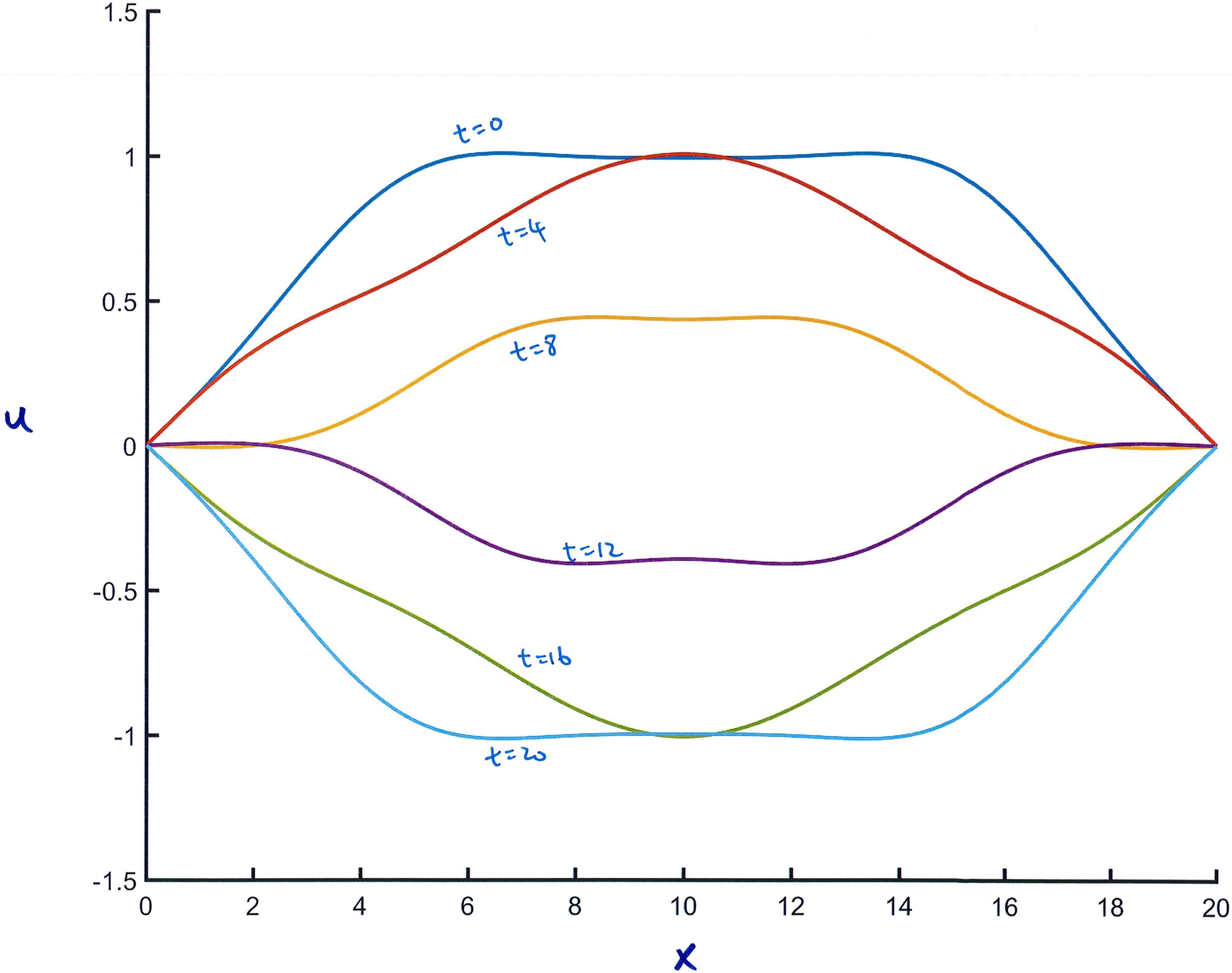
fundamental mode $(n=1)$

$$u_1(x,t)=\frac{\sqrt{2}\,8}{\pi^2}\cos\frac{\pi t}{20}\sin\frac{\pi x}{20}$$

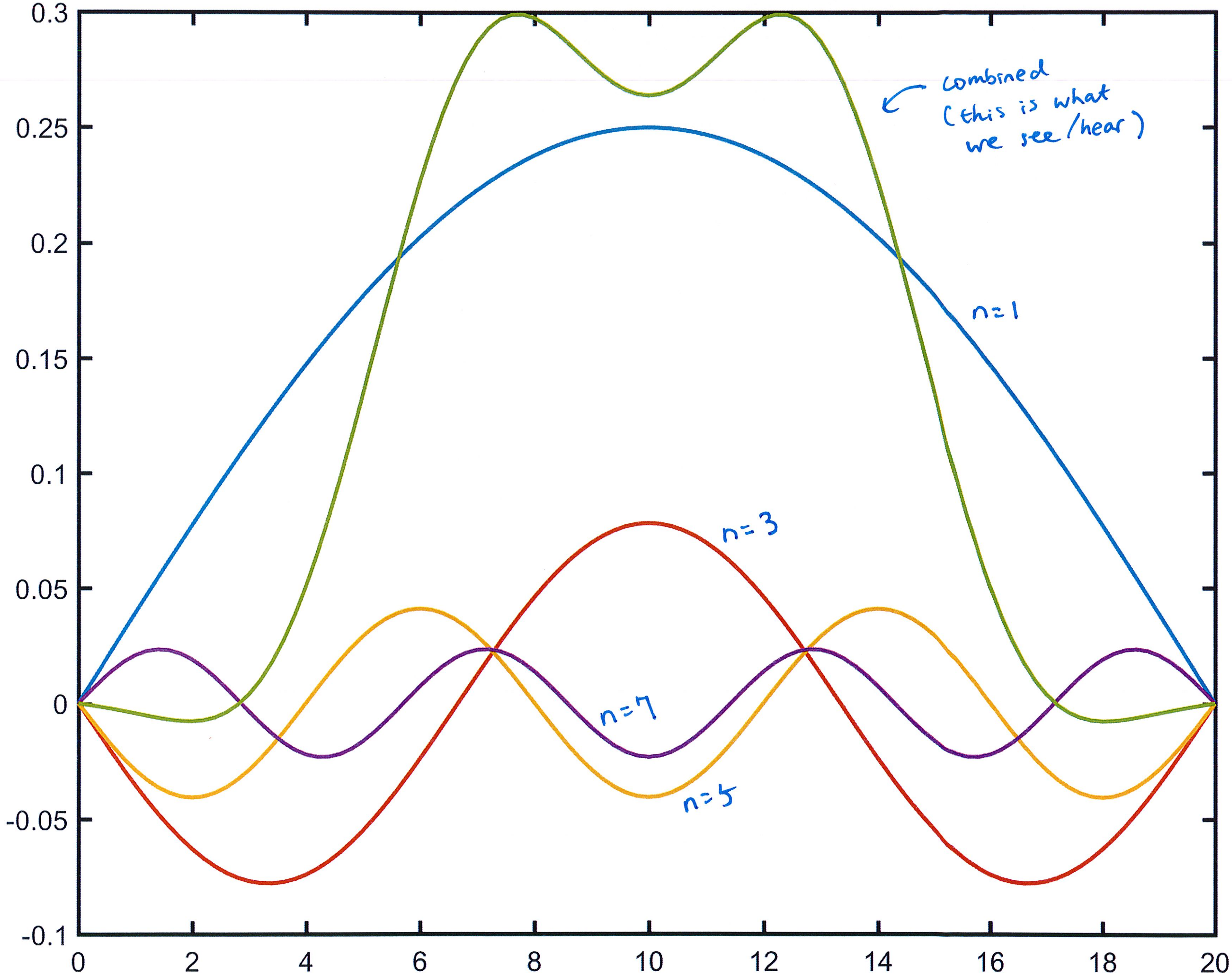
(this is dominant part, the part we mostly hear in an instrument)



1.5
1
0.5
0
-0.5
-1
-1.5
displacement
20
15
10
5
0
position
0
5
10
15
20
time



1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
14
16
18
20
u
x
t=0
t=4
t=8
t=12
t=16
t=20


0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
2
4
6
8
10
12
14
16
18
20
combined (this is what we see/hear)
n=1
n=3
n=7
n=5

Quiz 9

1. Use the method of separation of variables to replace the given partial differential equation with a pair of ordinary differential equations:

$$u_{xx} + u_{xt} + u_t = 0$$

2. What is the steady state solution of

$$u_{xx} = u_t, \quad 0 \le x \le 1, \quad t > 0$$
$$u(0,t) = 0, \quad u(1,t) = 2$$
$$u(x,0) = 2x + \sin \pi x$$