10.7 Wave Equation (continued)

$a^2 u_{xx} = u_{tt} \qquad 0 \le x \le L$

$u(0,t) = u(L,t) = 0$ (Dirichlet conditions)

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Case 1: $f(x) \neq 0, \; g(x) = 0$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Case 2: $f(x)=0,\ g(x)\neq 0$

$$u(x,t) = X(x)\,T(t)$$

$$u_{xx} = X''T \qquad u_{tt} = XT''$$

$$u(0,t)=0 \rightarrow X(0)=0$$

$$u(L,t)=0 \rightarrow X(L)=0$$

$$u(x,0)=0 \rightarrow X(x)T(0)=0 \rightarrow T(0)=0$$ ($T'(0)=0$ in case 1)

$X(x)$ solution is the same as case 1

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$X_n = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

T eq. after separation

$$T'' + a^2\lambda T = 0$$

$$T'' + \frac{a^2n^2\pi^2}{L^2}T = 0 \qquad T(0)=0$$

$$T(t) = k_1 \cos\frac{n\pi a t}{L} + k_2 \sin\frac{n\pi a t}{L}$$

$$T(0) = k_1 = 0$$

so $T_n = \sin\frac{n\pi a t}{L}$ case 1: $T_n = \cos\frac{n\pi a t}{L}$

fundamental solution:

$$u_n(x,t) = \sin\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}$$

general solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}}$$

IC: $u_t(x,0) = g(x)$

$$u_t(x,t) = \sum_{n=1}^{\infty} c_n \cdot \frac{n\pi a}{L} \cos\frac{n\pi a t}{L} \sin\frac{n\pi x}{L}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \underbrace{\left(c_n \cdot \frac{n\pi a}{L}\right)}_{\text{coeff. of Fourier Sine series}} \sin\frac{n\pi x}{L} = g(x)$$

$$\boxed{C_n \cdot \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

or

$$\boxed{C_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx}$$

solution will look like that of case 1

Case 3: $f(x) \neq 0,\ g(x) \neq 0$

the partial differential eq. $a^2 u_{xx} = u_{tt}$ is linear

if $v(x,t)$ is solution and $w(x,t)$ is solution

then $u(x,t) = v(x,t) + w(x,t)$ is also a solution

$$u(x,t) = \underbrace{v(x,t)}_{\text{solution of case 1}} + \underbrace{w(x,t)}_{\text{solution of case 2}}$$

so $a^2 v_{xx} = v_{tt}$

$a^2 w_{xx} = w_{tt}$

$$u_{xx} = v_{xx} + w_{xx}$$

$$u_{tt} = v_{tt} + w_{tt}$$

plug into $a^2 u_{xx} = u_{tt}$ to check that $u(x,t)$ satisfies the eq.

$$a^2 v_{xx} + a^2 w_{xx} = v_{tt} + w_{tt}$$

$$\underbrace{a^2 v_{xx} - v_{tt}}_{0} = -(\underbrace{a^2 w_{xx} - w_{tt}}_{0})$$

because $a^2 v_{xx} = v_{tt}$

because $a^2 w_{xx} = w_{tt}$

so $u(x,t) = v(x,t) + w(x,t)$ is a solution

so case 3 solution is

$$u(x,t) = \underbrace{\left(\sum_{n=1}^{\infty} c_n \cos\frac{n\pi a t}{L}\sin\frac{n\pi x}{L}\right)}_{\substack{\text{case 1} \\ (g(x)=0)}} + \underbrace{\left(\sum_{n=1}^{\infty} d_n \sin\frac{n\pi a t}{L}\sin\frac{n\pi x}{L}\right)}_{\substack{\text{case 2} \\ (f(x)=0)}}$$

$$\boxed{\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\ d_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$

Solution of wave equation describes <u>standing waves</u>

(waves not going anywhere)

but these standing waves can be modeled as combination of two traveling ~~ways~~ waves, one going to right, one to left.

they bounce off ends, go back to middle past each other, then bounced off ends, back to middle and so on.

what we see is due to interferences w/ each other.

$a^2 u_{xx} = u_{tt}$ 1-D (string)

$a^2(u_{xx} + u_{yy}) = u_{tt}$ 2-D (plate/membrane)

displacement: $u(x, y, t)$

separation: $u = X(x)\, Y(y)\, T(t)$