# 10.8 Laplace's Equation (part 1)

homework for 10.8 due Monday, 7/31.

1-D Heat eq. $\alpha^2 u_{xx} = u_t$

1-D Wave eq. $a^2 u_{xx} = u_{tt}$

(2-D wave eq. $a^2(u_{xx} + u_{yy}) = u_{tt}$ )

the left side of these can be expressed using the Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \cdots$

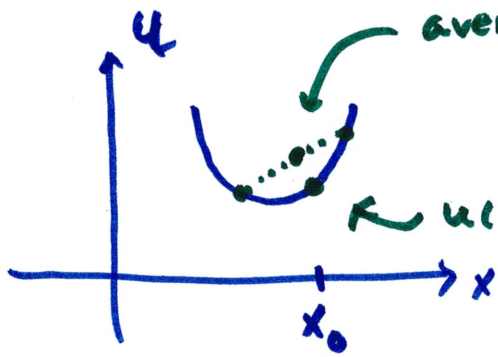
(space coords. only)

Heat eq: $\alpha^2 \nabla^2 u = u_t$

Wave eq: $a^2 \nabla^2 u = u_{tt}$

Physical meaning of $\nabla^2 u$ :

(1-D) $\qquad \nabla^2 u = u_{xx}$




$u_{xx} > 0$

u($x_0$) is less than average of u(x) nearby

if $u_{xx} < 0$, u($x_0$) is greater than average of u(x) nearby

Heat : $\alpha^2 \nabla^2 u = u_t$

if $\nabla^2 u > 0$, then u($x_0$) ← temp. less than avg temp. around it, so heat flows toward $x_0$ → $u_t > 0$ at $x_0$

Laplaces' Equation: $\nabla^2 u = 0$ no $\frac{\partial}{\partial t}$ in this eq.

we want to find $u$ such that at any $x$ the value of $u$ is <u>equal to</u> the avg. $u$ nearby.

→ $\nabla^2 u = 0$ can be interpreted as steady state solution of heat eq. (straight line solution).

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<u>2-D Laplace's Equation</u> (steady state heat of plate)

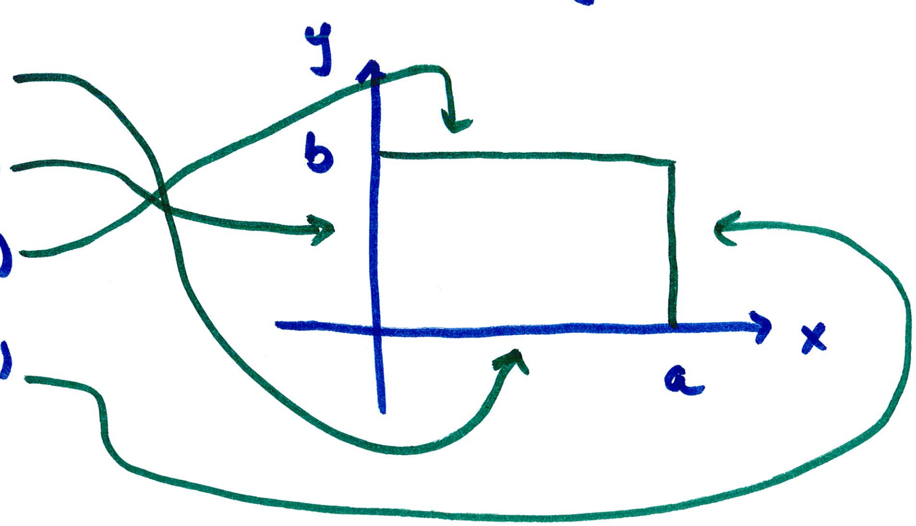
$$u_{xx} + u_{yy} = 0 \qquad a \le x \le a \qquad 0 \le y \le b$$

BC's: $u(x,0) = f_1(x)$

$u(0,y) = f_2(y)$

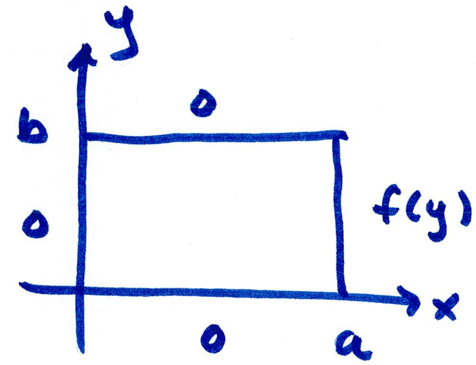
$u(x,b) = f_3(x)$

$u(a,y) = f_4(y)$



if all 4 BC's are zero, $u=0$ (trivial solution)

let's set 3 of them to zero




$$u_{xx}+u_{yy}=0$$

$$u(x,0)=0$$

$$u(0,y)=0$$

$$u(x,b)=0$$

$$u(a,y)=f(y)$$

$$u(x,y)=X(x)Y(y) \quad \text{(no } t\text{)}$$

$$X''Y+XY''=0$$

$$\frac{X''}{X}=-\frac{Y''}{Y}=\lambda$$

separation constant (why not $-\lambda$?, see next few steps)

$$X(x)Y(0)=0 \rightarrow Y(0)=0$$

$$X(0)Y(y)=0 \rightarrow X(0)=0$$

$$X(x)Y(b)=0 \rightarrow Y(b)=0$$

can't use last BC yet

$X'' - \lambda X = 0$ $\qquad X(0) = 0$

$Y'' + \lambda Y = 0$ $\qquad Y(0) = 0 \quad Y(b) = 0$

solve Y first, because we have 2 BC's

$\lambda_n = \left(\frac{n\pi}{b}\right)^2$

"L" for y ( $0 \le y \le b$ )

$$\boxed{Y_n = \sin\left(\frac{n\pi y}{b}\right)} \qquad n = 1, 2, 3, \ldots$$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0 \qquad X(0) = 0$$

$$X(x) = d_1 e^{\frac{n\pi x}{b}} + d_2 e^{-\frac{n\pi x}{b}}$$

usually written as

$$X(x) = k_1 \cosh\left(\frac{n\pi x}{b}\right) + k_2 \sinh\left(\frac{n\pi x}{b}\right)$$

$$X(0) = 0 = k_1$$

$$\boxed{X_n = \sinh\left(\frac{n\pi x}{b}\right)}$$

$$u_n = \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

general solution:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)}$$

last BC: $u(a,y) = f(y)$

$$f(y) = \sum_{n=1}^{\infty}\left[C_n \sinh\left(\frac{n\pi a}{b}\right)\right]\sin\left(\frac{n\pi y}{b}\right) \qquad 0 \le y \le b$$

Fourier sine series w/ $C_n \sinh\left(\frac{n\pi a}{b}\right)$ as coefficients

$$C_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

$$\boxed{C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{b} \boxed{\frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)}} \left[\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy\right] \sin\left(\frac{n\pi y}{b}\right)$$

behave like $e^{+\frac{n\pi(x-a)}{b}}$ (negative), so the series will converge rapidly (like heat eq., unlike wave eq.)

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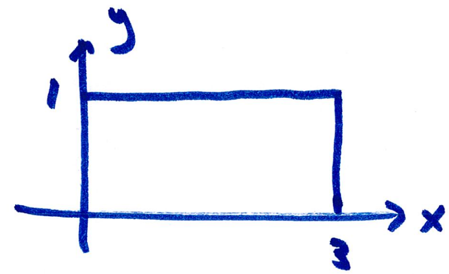
<u>example</u> $a=3, \; b=1$

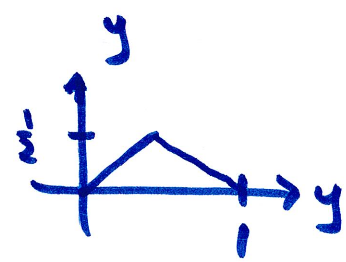



$u(x,0)=0$

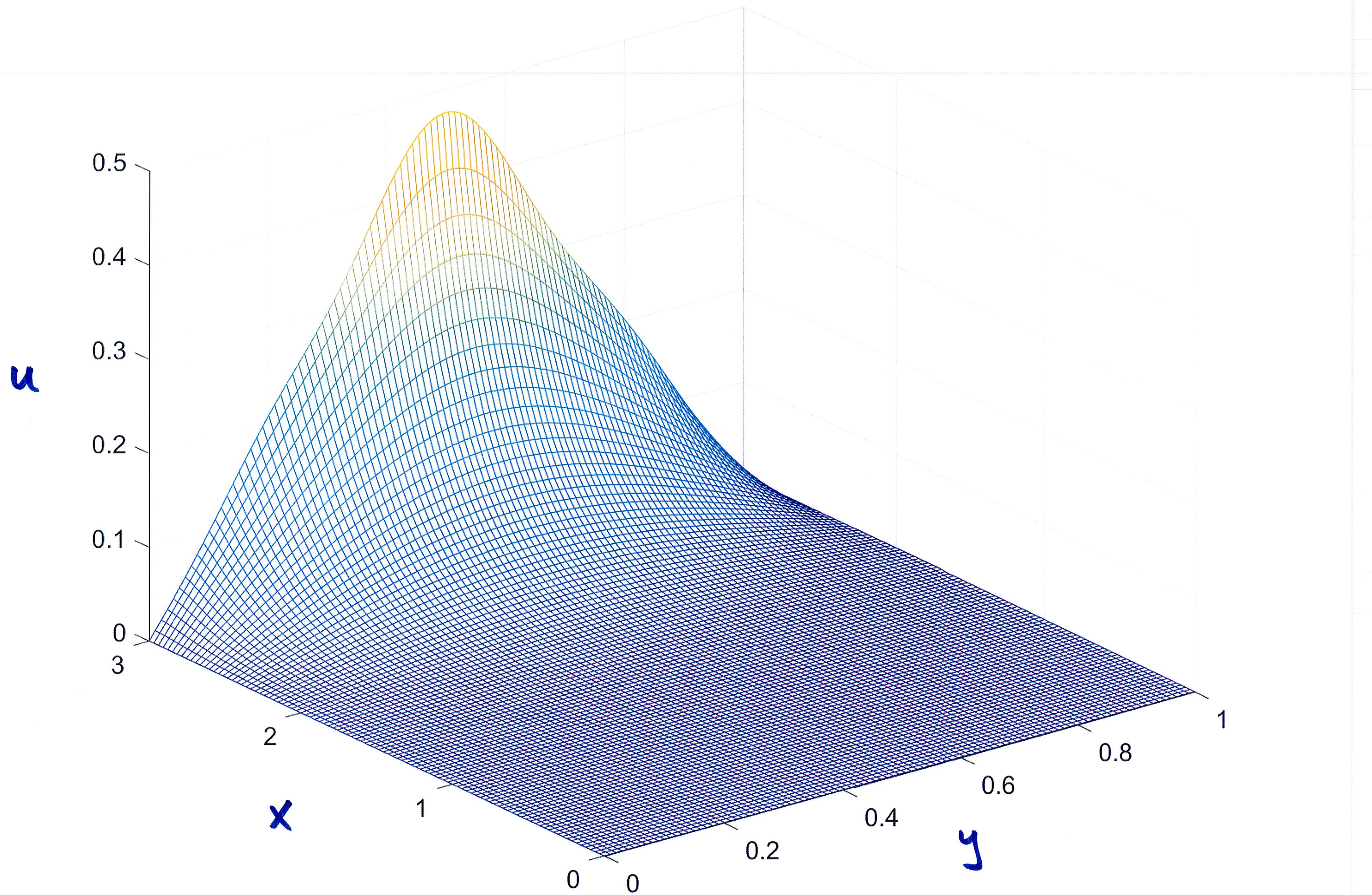
$u(0,y)=0$

$u(1,y)=0$

$$u(3,y) = \begin{cases} y & 0 \le y \le \frac{1}{2} \\ 1-y & \frac{1}{2} \le y \le 1 \end{cases} \rightarrow$$




$$u(x,y) = \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2 \sinh(3\pi)} \sinh(n\pi x) \sin(n\pi y)$$


0.5
0.4
0.3
0.2
0.1
0
u
3
2
1
0
x
0
0.2
0.4
0.6
0.8
1
y