10.8 Laplace’s Equation (part 1)

homework for 10.8 due Monday, 7/31.

1-D Heat eq. \( \alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \)

1-D Wave eq. \( \alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \)

(2-D wave eq. \( \alpha^2 (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = \frac{\partial^2 u}{\partial t^2} \))

the left side of these can be expressed using

the Laplacian operator \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \ldots \)

(space coords. only)

a Heat eq.: \( \alpha^2 \nabla^2 u = \frac{\partial u}{\partial t} \)

Wave eq.: \( \alpha^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2} \)
Physical meaning of $\nabla^2 u$:

\[(1-0) \quad \nabla^2 u = u_{xx}\]

\[\text{average of } u(x) \text{ near } u(x_0)\]

if $u_{xx} > 0$, $u(x_0)$ is greater than average of $u(x)$ nearby

if $u_{xx} < 0$, $u(x_0)$ is less than average of $u(x)$ nearby

Hech: $\alpha^2 \nabla^2 u = u_t$}

if $\nabla^2 u > 0$, then $u(x_0)$ less than temp. around it, so heat flows toward $x_0 \Rightarrow u_t > 0$ at $x_0$
Laplace's Equation: \( \nabla^2 u = 0 \) no \( \frac{\partial}{\partial t} \) in this eq.

we want to find \( u \) such that

at any \( x \) the value of \( u \) is equal to

the avg. \( u \) nearby.

\[ \nabla^2 u = 0 \] can be interpreted as

steady state solution of heat eq.

(straight line solution).

2-D Laplace's Equation

\[ u_{xx} + u_{yy} = 0 \quad a \leq x \leq a \quad 0 \leq y \leq b \]

BC's: \( u(x,0) = f_1(x) \)
\( u(0,y) = f_2(y) \)
\( u(x,b) = f_3(x) \)
\( u(a,y) = f_4(y) \)
if all 4 BC's are zero, $u = 0$ (trivial solution)

let's set 3 of them to zero

$u_{xx} + u_{yy} = 0$
$u(x, 0) = 0$
$u(0, y) = 0$
$u(x, b) = 0$
$u(a, y) = f(y)$

$u(x, y) = X(x)Y(y)$  \(\text{(no } t\text{)}\)

$X''Y + XY'' = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

$X(x)Y(0) = 0 \rightarrow Y(0) = 0$
$X(0)Y(y) = 0 \rightarrow X(0) = 0$
$X(x)Y(b) = 0 \rightarrow Y(b) = 0$
$X(0)Y(b) = 0 \rightarrow Y(0) = 0$

separation constant

(why not $-\lambda$? see next few steps)

Can't use last BC yet
\( X'' - \lambda X = 0 \quad X(0) = 0 \)

\( Y'' + \lambda Y = 0 \quad Y(0) = 0 \quad Y(b) = 0 \)

Solve \( Y \) first, because we have 2 BC's

\( \lambda_n = \left( \frac{n\pi}{b} \right)^2 \)

\( Y_n = \sin \left( \frac{n\pi y}{b} \right) \quad n = 1, 2, 3, \ldots \)

\( X'' - \frac{n^2\pi^2}{b^2} X = 0 \quad X(0) = 0 \)

\[ X(x) = d_1 e^{\frac{n\pi x}{b}} + d_2 e^{-\frac{n\pi x}{b}} \]

Usually written as

\[ X(x) = k_1 \cosh \left( \frac{n\pi x}{b} \right) + k_2 \sinh \left( \frac{n\pi x}{b} \right) \]

\( X(0) = 0 = k_1 \)

\[ X_n = \sinh \left( \frac{n\pi x}{b} \right) \]
\[ u_n = \sinh \left( \frac{n\pi x}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \]

**General Solution:**
\[ u(x, y) = \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi x}{b} \right) \sin \left( \frac{n\pi y}{b} \right) \]

**Last BC:** \( u(a, y) = f(y) \)
\[ f(y) = \sum_{n=1}^{\infty} \left[ C_n \sinh \left( \frac{n\pi a}{b} \right) \right] \sin \left( \frac{n\pi y}{b} \right) \quad 0 \leq y \leq b \]

**Fourier Sine Series**
with \( C_n \sinh \left( \frac{n\pi a}{b} \right) \) as coefficients

\[ C_n \sinh \left( \frac{n\pi a}{b} \right) = \frac{2}{b} \int_0^b f(y) \sin \left( \frac{n\pi y}{b} \right) \, dy \]

\[ C_n = \frac{2}{b \sinh \left( \frac{n\pi a}{b} \right)} \int_0^b f(y) \sin \left( \frac{n\pi y}{b} \right) \, dy \]
\[
\begin{align*}
  u(x, y) = \sum_{n=1}^{\infty} \frac{2}{b} \frac{\sinh \left( \frac{n \pi x}{b} \right)}{\sinh \left( \frac{n \pi a}{b} \right)} \left[ \int_0^b f(y) \sin \left( \frac{n \pi y}{b} \right) dy \right] \sin \left( \frac{n \pi y}{b} \right)
  \end{align*}
\]

\[\text{(negative)} \quad \frac{n \pi (x-a)}{b}\]

behave like \(e\), so the series will converge rapidly (like heat e.g., unlike wave e.g.)

**Example**

\(a = 3, \ b = 1\)

\[
\begin{align*}
  u(x, 0) &= 0 \\
  u(0, y) &= 0 \\
  u(1, y) &= 0 \quad 0 \leq y \leq \frac{1}{2} \\
  u(3, y) &= 1 - y \quad \frac{1}{2} \leq y \leq 1 \\
  u(x, y) &= \sum_{n=1}^{\infty} \frac{4 \sin \left( \frac{n \pi y}{2} \right)}{n^2 \pi^2 \sinh (3\pi)} \sinh \left( \frac{n \pi x}{3} \right) \sin \left( \frac{n \pi y}{3} \right)
\end{align*}
\]