5.3 Series Solutions near an Ordinary Pt (part 2)

recall, from last time, $a_0 = y(x_0)$

$a_1 = y'(x_0)$

how does a_n relate to $y^{(n)}(x_0)$?

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

solution:
$$y = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$
$$y' = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x_0)^3 + \cdots$$
$$y'' = 2a_2 + 3\cdot 2\cdot a_3(x-x_0) + 4\cdot 3\cdot a_4(x-x_0)^2 + 5\cdot 4\cdot a_5(x-x_0)^3 + \cdots$$
$$y''' = 3\cdot 2\cdot a_3 \cancel{(x-x_0} + 4\cdot 3\cdot 2\cdot a_4(x-x_0) + \cdots$$
$$y^{(4)} = 4\cdot 3\cdot 2\cdot a_4 + \cdots$$

at $x = x_0$

$y(x_0) = a_0 \quad = 0!\,a_0$

$y'(x_0) = a_1 \quad = 1!\,a_1$

$y''(x_0) = 2a_2 \quad = 2!\,a_2$

$y'''(x_0) = 3\cdot 2\cdot a_3 \quad = 3!\,a_3$

$y^{(4)}(x_0) = 4\cdot 3\cdot 2\cdot a_4 \quad = 4!\,a_4$

$$\boxed{y^{(n)}(x_0) = n!\,a_n}$$

can use this to find a_n by successive differentiation

example $\quad y'' + xy' + y = 0 \qquad y(0) = 0,\ y'(0) = 1$

find a_3 (this also tells us $y'''(x_0)$)

from $y^{(n)}(x_0) = n!\,a_n$ and $x_0 = 0$

$y(0) = 0!\,a_0 = 0 \rightarrow a_0 = 0$

$y'(0) = 1!\,a_1 = 1 \rightarrow a_1 = 1$

$y'' = -xy' - y$

$y''(0) = -y(0) = 2!\,a_2 \rightarrow a_2 = 0$

$$y'' = -xy' - y$$

$$y''' = -xy'' - y' - y' = -xy'' - 2y'$$

$$y'''(0) = -2y'(0) = -2 = 3!\, a_3 \rightarrow \boxed{a_3 = -\frac{1}{3}}$$

this allows us to find as many a_n's as we want w/o dealing w/ series directly (no recurrence relation)

w/o series, how to find radius of convergence?

(no ratio test)

but we can use a result from complex analysis to find radius of convergence.

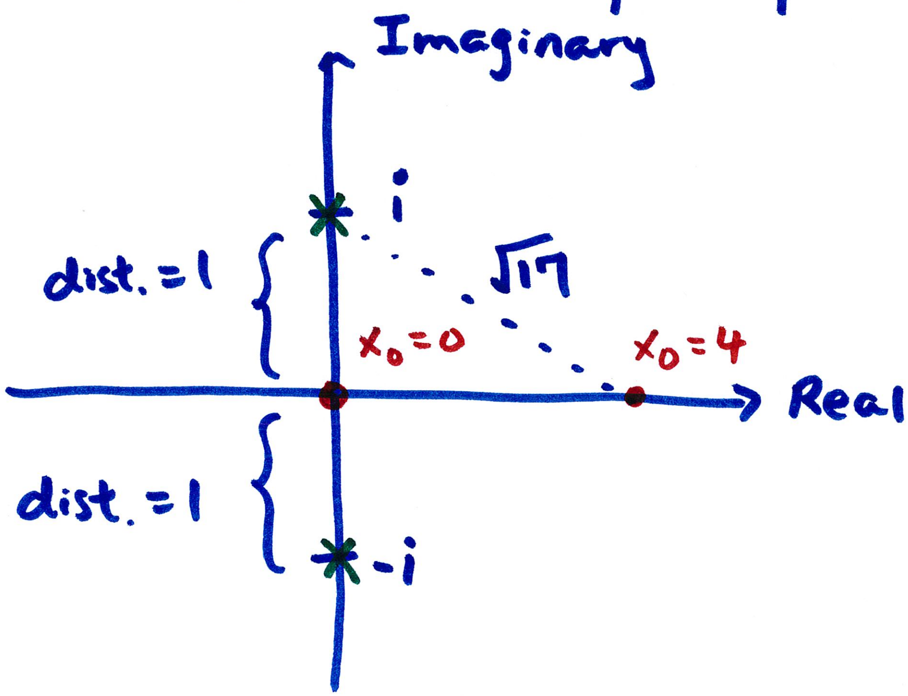
<u>example</u> $(1+x^2)y''+4xy'+y=0 \qquad x_0=0$

$$y''+\frac{4x}{1+x^2}y'+\frac{1}{1+x^2}y=0$$

the poles of $\frac{4x}{1+x^2}$ and $\frac{1}{1+x^2}$

are $1+x^2=0 \rightarrow x^2=-1 \rightarrow x=i,-i$

put on complex plane



the distance from x_0 to the nearest pole is the <u>lower bound</u> of radius of convergence

for $x_0=0$, radius of conv. ≥ 1

for $x_0=4$, radius of conv. $\geq \sqrt{17}$

DE's with difficult coefficients

example $\qquad e^x y'' + xy = 0 \qquad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$e^x \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

easy.

need Taylor series for e^x near $x_0 = 0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

in practice, truncate the Taylor series

if we stay near $x_0 = 0$, don't need too many terms

here, $e^x \approx 1 + x + \frac{x^2}{2!}$ is good enough

$$\left(1+x+\frac{x^2}{2}\right)\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}+\sum_{n=0}^{\infty} a_n x^{n+1}=0$$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}+\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-1}+\sum_{n=2}^{\infty} \frac{a_n(n)(n-1)}{2}x^{n}$$

$$+\sum_{n=0}^{\infty} a_n x^{n+1}=0$$

Shift ...

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^{n}+\sum_{n=1}^{\infty} a_{n+1}(n+1)(n)x^{n}+\sum_{n=2}^{\infty} \frac{a_n(n)(n-1)}{2}x^{n}$$

$$+\sum_{n=1}^{\infty} a_{n-1}x^{n}=0$$

$n=0$: $\quad 2a_2=0 \rightarrow a_2=0$

$n=1$: $\quad 6a_3+2a_2+a_0=0 \rightarrow a_3=-\frac{1}{6}a_0$

$n=2$: $\quad 12a_4+6a_3+a_2+a_1=0 \rightarrow a_4=\frac{1}{12}a_0-\frac{1}{12}a_1$

$n=3$: $\quad 20a_5+12a_4+3a_3+a_2=0 \rightarrow a_5=-\frac{1}{40}a_0+\frac{1}{20}a_1$

Solution: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$y = a_0 + a_1 x + \underbrace{\left(-\tfrac{1}{6} a_0\right)}_{a_3} x^3 + \left(\tfrac{1}{12} a_0 - \tfrac{1}{12} a_1\right) x^4 + \left(-\tfrac{1}{40} a_0 + \tfrac{1}{20} a_1\right) x^5 + \cdots$$

$$= a_0 \left(1 - \tfrac{1}{6} x^3 + \tfrac{1}{12} x^4 - \tfrac{1}{40} x^5 + \cdots\right) + a_1 \left(x - \tfrac{1}{12} x^4 + \tfrac{1}{20} x^5 + \cdots\right)$$

fundamental solutions: $y_1 = 1 - \tfrac{1}{6} x^3 + \tfrac{1}{12} x^4 - \tfrac{1}{40} x^5 + \cdots$

$y_2 = x - \tfrac{1}{12} x^4 + \tfrac{1}{20} x^5 + \cdots$

on HW, want first 3 non-zero terms in each of y_1 and y_2 $\longrightarrow$ want 6 non-zero terms in y

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \underbrace{a_6 x^6 + a_7 x^7}_{\text{just in case}}$$

some may be zero