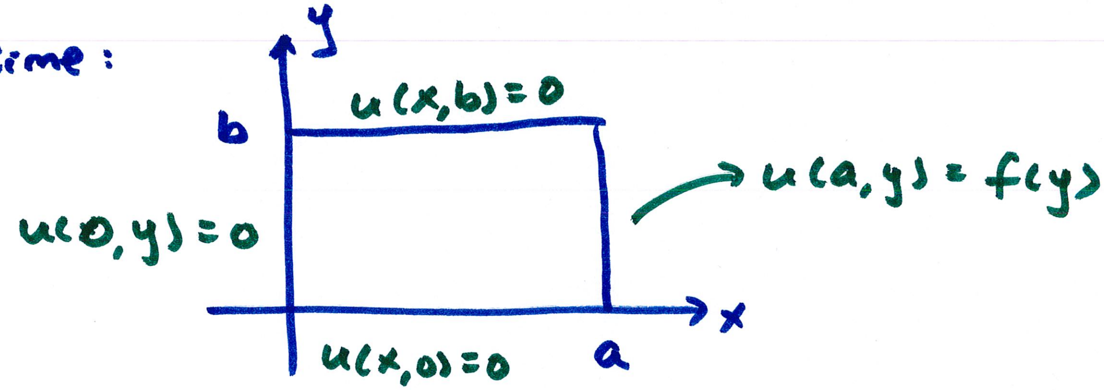
10.8 Laplace's Eq. (continued)

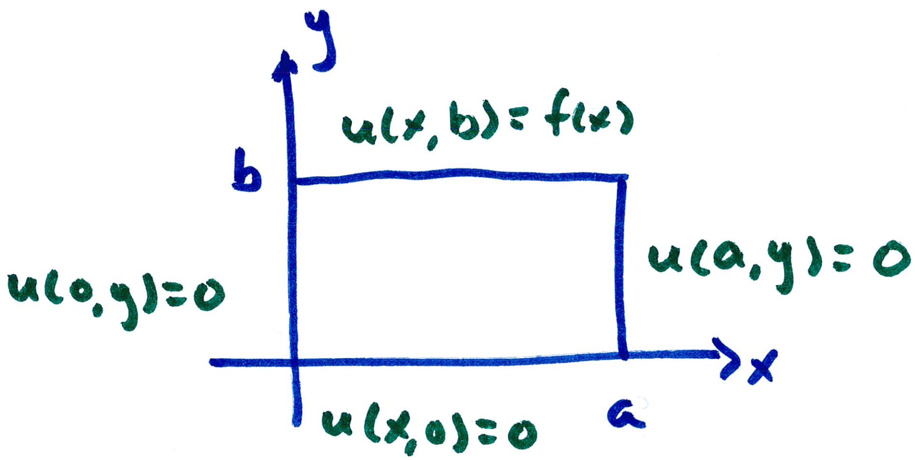
last time:



Solution: $u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$

$$C_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

now move nonhomogeneous BC

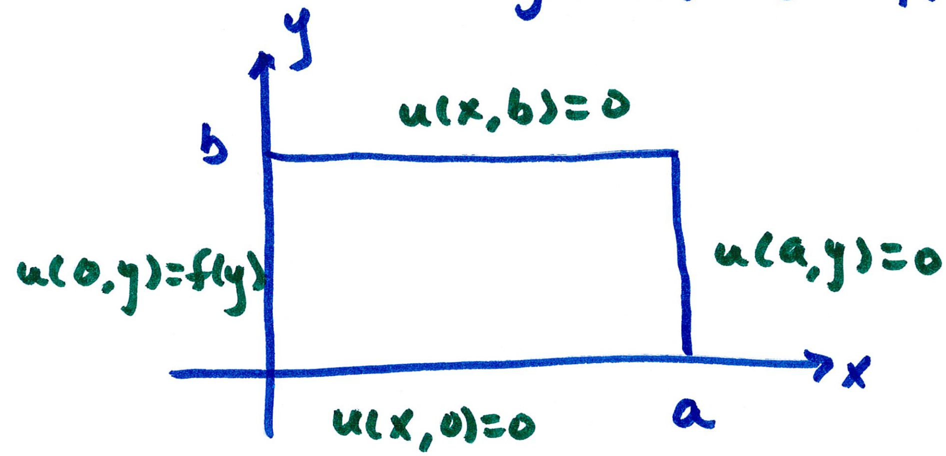


interchange x and y and a and b

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

now move nonhomogeneous to left edge



$$u_{xx} + u_{yy} = 0$$

Y has enough BC's to solve for (pos. eigenvalues)

BC's: $\boxed{\begin{array}{ll} Y(0)=0 & \text{(bottom)} \\ Y(b)=0 & \text{(top)} \end{array}}$

$X(a) = 0$ (right)

$$u(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$Y'' + \lambda Y = 0 \qquad Y(0) = Y(b) = 0$$

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2 \qquad Y_n = \sin\left(\frac{n\pi y}{b}\right)$$

$$X'' - \lambda X = 0 \qquad X(a) = 0$$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0$$

$$X = k_1 \cosh\frac{n\pi x}{b} + k_2 \sinh\frac{n\pi x}{b}$$

$$0 = k_1 \underbrace{\cosh\frac{n\pi a}{b}}_{\neq 0} + k_2 \underbrace{\sinh\frac{n\pi a}{b}}_{\neq 0}$$

can't solve for k_1 or k_2 easily

$\rightarrow$ messy ~~fcn~~ eigenfunction

fix:

$$X = h_1 e^{-\frac{n\pi x}{b}} + h_2 e^{\frac{n\pi x}{b}} \qquad X(a) = 0$$

$$0 = h_1 e^{-n\pi a/b} + h_2 e^{n\pi a/b}$$

$$X = \underbrace{g_1 e^{\frac{n\pi a}{b}}}_{h_1} e^{-\frac{n\pi x}{b}} + \underbrace{g_2 e^{-\frac{n\pi a}{b}}}_{h_2} e^{\frac{n\pi x}{b}}$$

$$= g_1 e^{-\frac{n\pi}{b}(x-a)} + g_2 e^{\frac{n\pi}{b}(x-a)}$$

$$X = d_1 \cosh \frac{n\pi(x-a)}{b} + d_2 \sinh \frac{n\pi(x-a)}{b}$$

$$X(a) = 0$$

$$0 = d_1 \qquad X_n = \sinh \frac{n\pi(x-a)}{b}$$

$$u_n = \sinh \frac{n\pi(x-a)}{b} \sin \frac{n\pi y}{b}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi(x-a)}{b} \sin \frac{n\pi y}{b}}$$

$$u(0,y) = f(y)$$

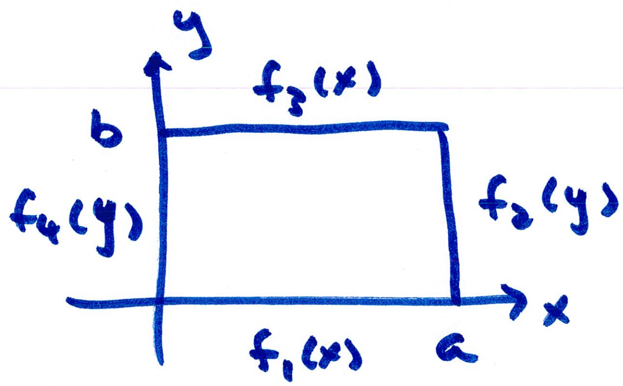
$$f(y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi(-a)}{b} \sin \frac{n\pi y}{b}$$

$$\boxed{C_n \sinh \frac{n\pi(-a)}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy}$$

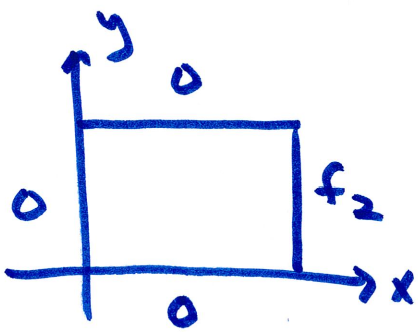
interchange x and y and a and b for
the case where nonhomogeneous BC at bottom

What about nonhomogeneous BC's all around?

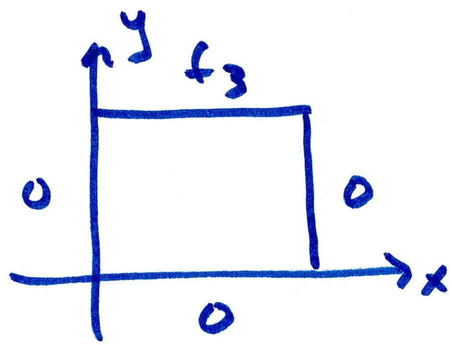
solution of



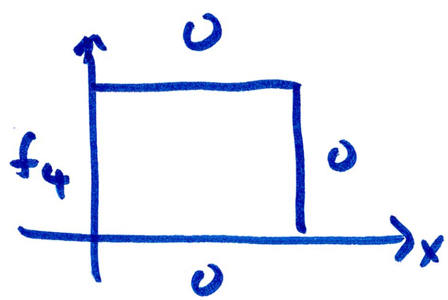
is a linear combination of



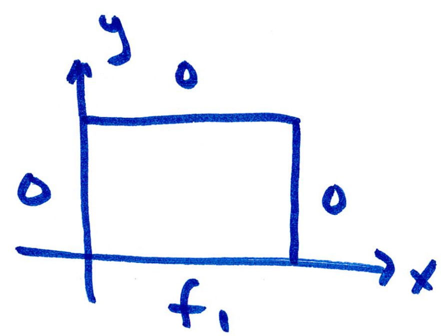
and



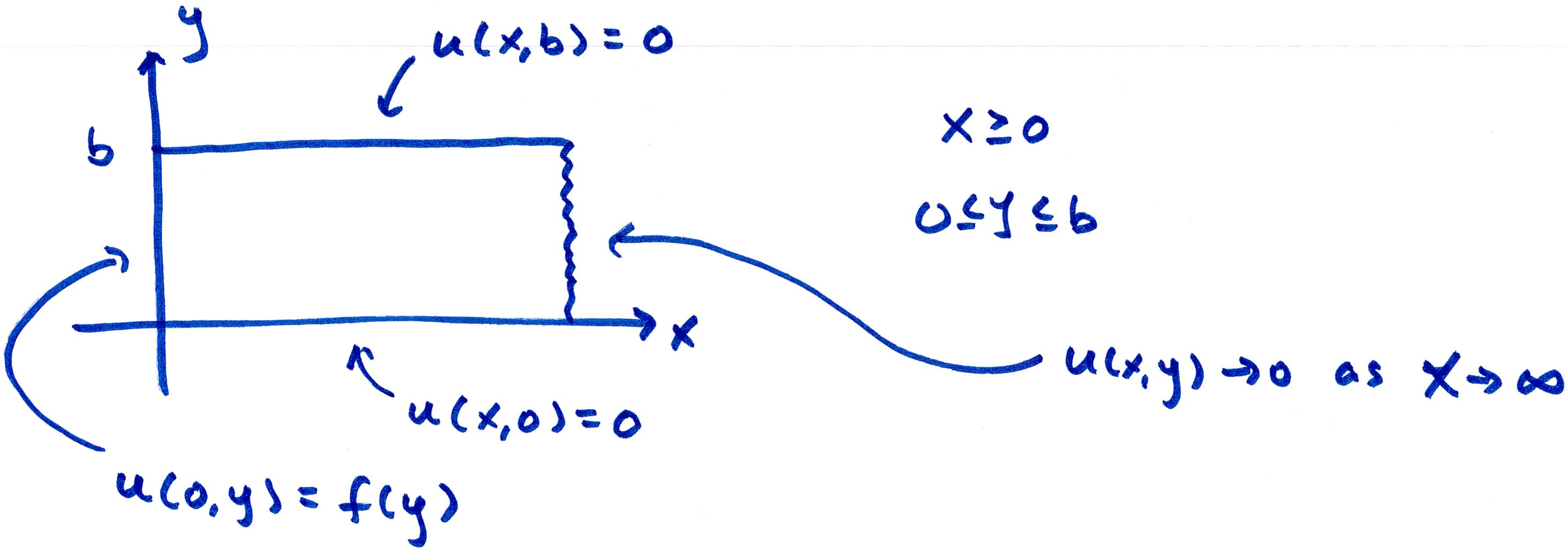
and



and



Semi-infinite Domain



$$u_{xx}+u_{yy}=0 \qquad u = XY$$

solve variable w/ homogeneous BC's first
(Y here)

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \qquad Y(0)=Y(b)=0$$

$$Y''+\lambda Y=0 \qquad \lambda_n = \left(\frac{n\pi}{b}\right)^2 \quad Y_n = \sin\left(\frac{n\pi y}{b}\right)$$

$$X'' - \frac{n^2\pi^2}{b^2} X = 0$$

"BC" : $u(x,y) \to 0$ as $\underbrace{x \to \infty}$

exponential better choice than hyperbolic functions

$$X = k_1 e^{\frac{n\pi x}{b}} + k_2 e^{-\frac{n\pi x}{b}}$$

$u \to 0$ as $x \to \infty$

$X(x)Y(y) \to 0$ as $x \to \infty$

$X \to 0$ as $x \to \infty$

$k_1 = 0 \qquad X_n = e^{-\frac{n\pi x}{b}}$

$$u_n = e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b}}$$

~~u(x,0) = f(x)~~ ~~etc~~ $u(0,y) = f(y)$

$$\boxed{c_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy}$$

basically the heat eq. solution

Quiz 10

1. Find a Fourier <u>sine</u> series representation of

$$f(x) = 2, \quad 0 < x < \pi, \text{ with odd extension}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

where

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx, \quad m = 0, 1, 2, \ldots$$

and

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx, \quad m = 1, 2, 3, \ldots$$

2. Solve the wave equation

$$u_{xx} = u_{tt}, \quad 0 \le x \le L, \ \ t > 0$$
$$u(0,t) = 0, \ \ u(L,t) = 0$$
$$u(x,0) = \sin \frac{\pi x}{L} + 0.5 \sin \frac{3\pi x}{L}$$