5.4 Euler Equations and Regular Singular Points

$P(x)y'' + Q(x)y' + R(x)y = 0$

if $P(x_0) \neq 0$, x_0 is an <u>ordinary point</u>

if $P(x_0) = 0$, x_0 is a <u>singular point</u>

there are two types of singular points

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

if $\boxed{\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}}$ <u>AND</u> $\boxed{\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}}$

are <u>both</u> finite, then x_0 is a <u>regular</u> <u>singular point</u>

else x_0 is an <u>irregular</u> <u>singular</u> <u>pt</u>

Example

$$x^2(1-x)y'' + (x-2)y' - 3xy = 0$$

$$y'' + \frac{x-2}{x^2(1-x)}\,y' - \frac{3x}{x^2(1-x)}\,y = 0$$

singular pts: $x_0 = 0$, $x_0 = 1$

$x_0 = 0$

$$\lim_{x\to 0} x \cdot \frac{x-2}{x^2(1-x)} = \pm\infty \quad \text{(depending on which side of 0)}$$

$$\lim_{x\to 0} x^2 \cdot \frac{-3x}{x^2(1-x)} = -3$$

at least is NOT finite, so $x_0 = 0$ is an irregular singular pt

$x_0 = 1$

$$\lim_{x\to 1} (x-1)\frac{x-2}{x^2(1-x)} = -1$$

$$\lim_{x\to 1} (x-1)^2 \frac{-3x}{x^2(1-x)} = 0$$

so $x_0 = 1$ is a regular s.p.

the simplest case : <u>Euler equation</u>

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad \alpha, \beta \text{ are } \underline{\text{constants}}$$

$$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0 \qquad x=0 \text{ is a singular pt}$$

$$\lim_{x\to 0} x \cdot \frac{\alpha}{x} = \alpha \qquad \lim_{x\to 0} x^2 \cdot \frac{\beta}{x^2} = \beta$$

so $x=0$ is a <u>regular singular pt</u>

Fundamental solutions are NOT of the form $y = e^{rx}$

but are $\boxed{y = x^r}$ $(x \neq 0)$

$$y' = r x^{r-1} \qquad y'' = (r)(r-1) x^{r-2}$$

plug into $x^2 y'' + \alpha x y' + \beta y = 0$

$$x^2 (r)(r-1) x^{r-2} + \alpha x (r) x^{r-1} + \beta x^r = 0$$

$$(r)(r-1) x^r + \alpha (r) x^r + \beta x^r = 0$$

$$[(r)(r-1) + \alpha (r) + \beta] x^r = 0$$

not zero (since $x \neq 0$)

$$\boxed{(r)(r-1) + \alpha r + \beta = 0}$$

solve for the two roots

indicial equation (similar to characteristic equation)

if roots are distinct ($r_1 \neq r_2$)

general solution: $\boxed{y = c_1 |x|^{r_1} + c_2 |x|^{r_2}}$

if roots are repeated ($r_1 = r_2 = r$)

general solution: $\boxed{y = c_1 |x|^r + c_2 |x|^r \ln|x|}$

if roots are complex conjugates ($r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$)

general solution: $\boxed{y = c_1 |x|^\lambda \cos(\mu \ln|x|) + c_2 |x|^\lambda \sin(\mu \ln|x|)}$

<u>example</u> $6x^2y'' + 5xy' - y = 0$

sub into DE: $y = x^r$, $y' = rx^{r-1}$, $y'' = (r)(r-1)x^{r-2}$

$$6(r)(r-1)x^r + 5r\,x^r - x^r = 0$$

$$x^r\left[\underbrace{6(r)(r-1) + 5r - 1}\right] = 0$$

$$6r^2 - r - 1 = 0 \qquad (2r-1)(3r+1) = 0$$

$$r = 1/2, \quad r = -1/3$$

gen. solution: $y = c_1|x|^{1/2} + c_2|x|^{-1/3}$

let $y(1) = 1$ and $y'(1) = 0$

$x > 0$ $\longrightarrow$ ok to drop absolute value

$$\left.\begin{aligned} y = c_1x^{1/2} + c_2x^{-1/3} &\longrightarrow 1 = c_1 + c_2 \\ y' = \tfrac{1}{2}c_1x^{-1/2} - \tfrac{1}{3}c_2x^{-4/3} &\longrightarrow 0 = \tfrac{1}{2}c_1 - \tfrac{1}{3}c_2 \\ &\ 0 = 3c_1 - 2c_2 \end{aligned}\right\} \begin{aligned} &\cancel{c_1 = \tfrac{1}{2}} \\ &\cancel{c_2 = \tfrac{1}{2}} \end{aligned}$$

~~solution:~~ $\cancel{y = \frac{1}{2}x^{1/2} + \frac{1}{2}x^{1/3}}$

$c_1 + c_2 = 1 \quad \longrightarrow \quad 2c_1 + 2c_2 = 2$

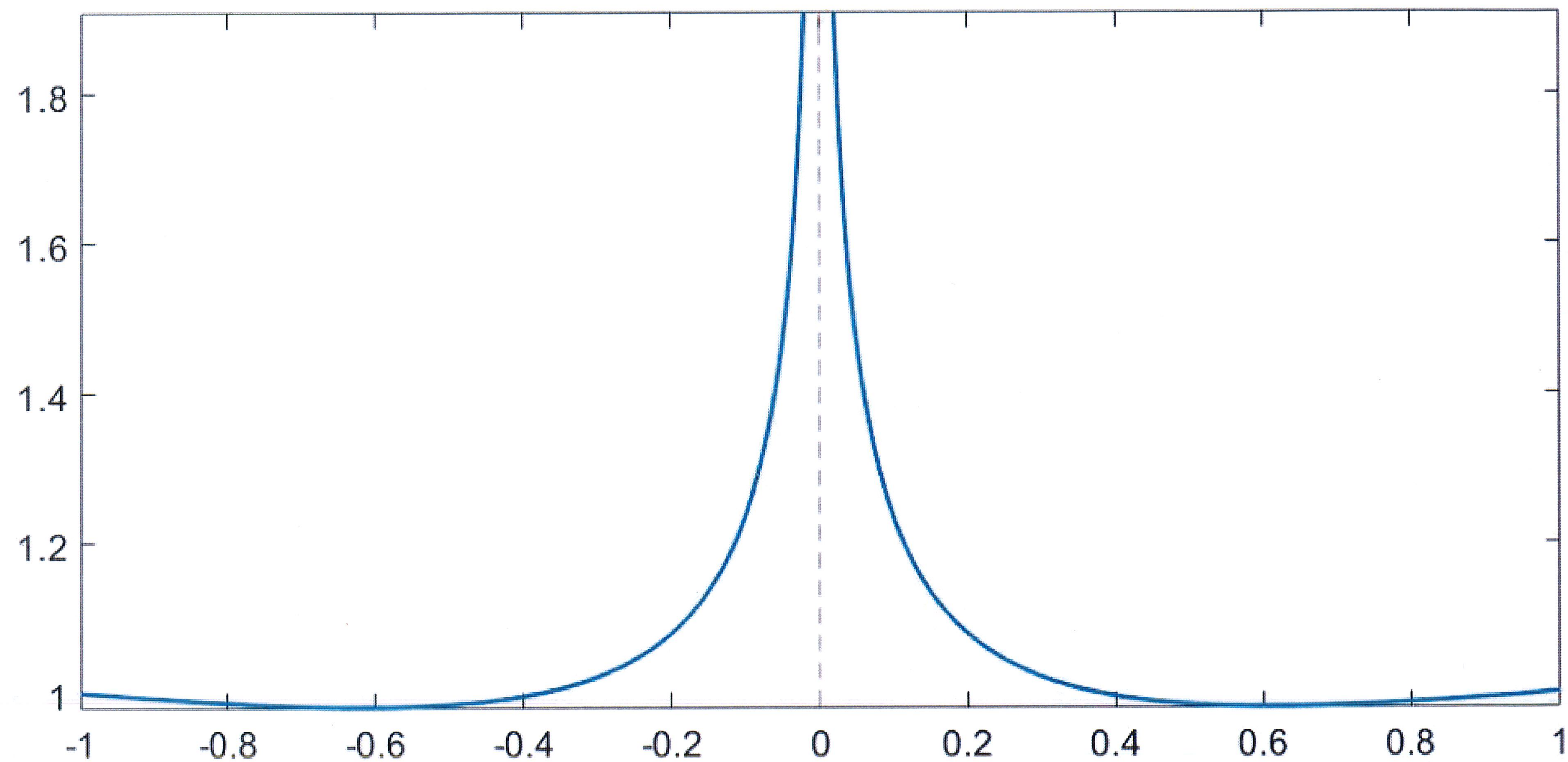
$3c_1 - 2c_2 = 0$

$5c_1 = 2 \qquad c_1 = \frac{2}{5}$

$c_2 = \frac{3}{5}$

solution: $y = \frac{2}{5}|x|^{1/2} + \frac{3}{5}|x|^{-1/3}$

$x = 0$ is singular pt



1.8
1.6
1.4
1.2
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

example $x^2y'' + 3xy' + 2y = 0$

$x^2y'' + \alpha xy' + \beta y = 0$

$r(r-1) + \alpha r + \beta = 0$

$r(r-1) + 3r + 2 = 0$

$r^2 + 2r + 2 = 0$

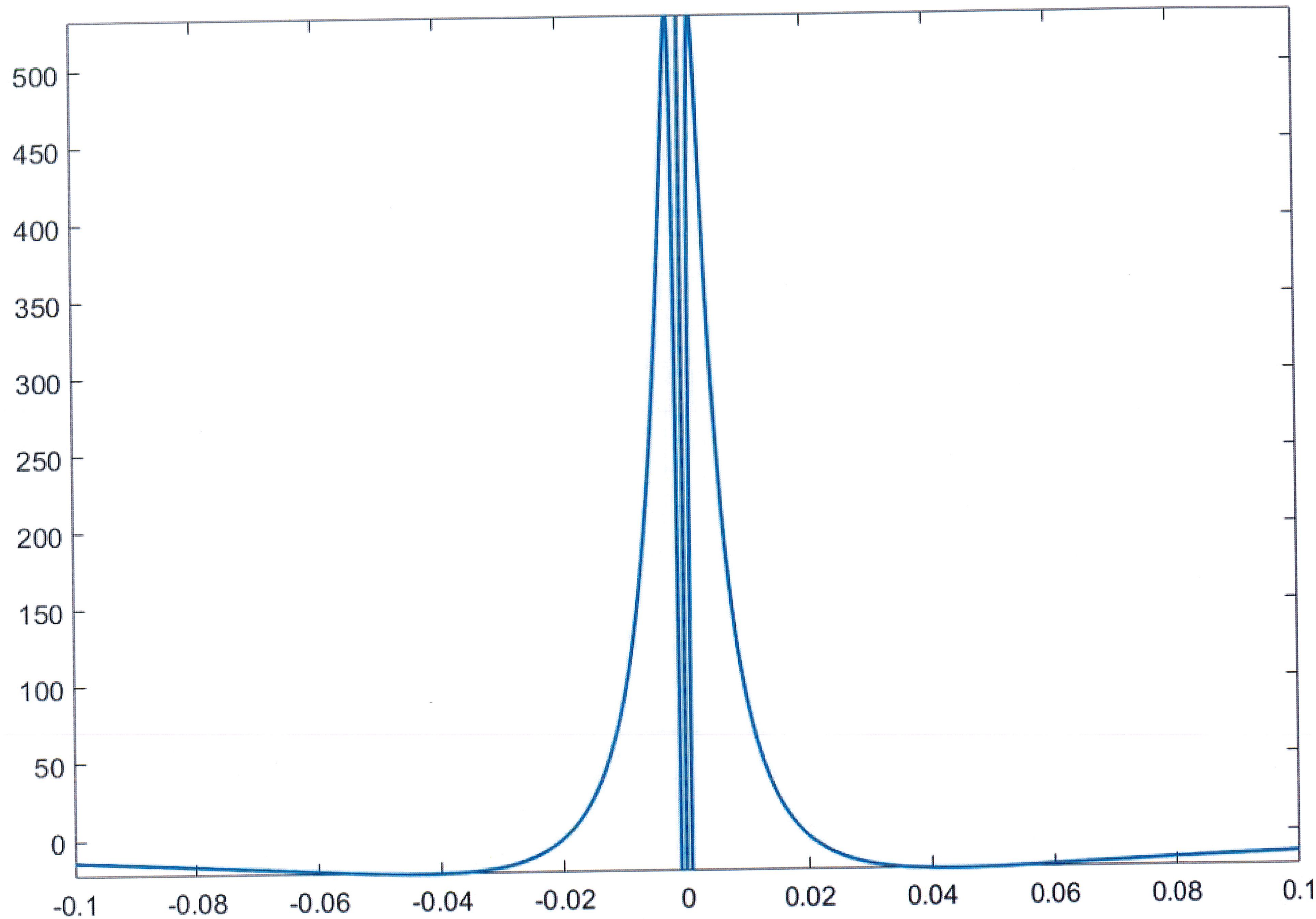
$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Solution: $y = C_1 |x|^{-1}(\cos(\ln|x|))$

$+ C_2 |x|^{-1}(\sin(\ln|x|))$

as $x \to 0$, $y \to \pm\infty$ (depending on C_1, C_2)

oscillations



500
450
400
350
300
250
200
150
100
50
0
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1

Quiz 1

1. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$$

2. Find the recurrence relation and the first 3 non-zero terms in the series solution to

$$y'' - xy' = 0, \qquad x_0 = 0$$