# 5.5 Series Solutions Near Regular Singular Point (part 1)

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$$

if $P(x_0) = 0$

but both $\displaystyle\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}$ and

$\displaystyle\lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)}$ are finite

$x_0$ is a regular singular point (rsp)

$$y'' + \frac{Q(x)}{P(x)}\,y' + \frac{R(x)}{P(x)}\,y = 0$$

let $\dfrac{Q}{P} = p$ (lower case p)

$\dfrac{R}{P} = q$ (lower case q)

$$y'' + p(x)\,y' + q(x)\,y = 0$$

for convenience, let $x_0 = 0$

if $x_0 = 0$ is rsp, then

$$x^2 y'' + x \underbrace{[x\,p(x)]}\, y' + \underbrace{[x^2 g(x)]}\, y = 0$$

are both analytic ( can be expressed as power series)

$$x\,p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$$

$$x^2 g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

if all $p_n$ and $g_n$ are zero except $p_0$ and $g_0$

$\rightarrow$ Euler equation $\quad x^2 y'' + x p_0 y' + g_0 y = 0$

$$x^2 y'' + \alpha x y' + \beta y = 0$$

if not constants

then series solutions for $y$

extend Euler eq solutions : $x^r$

if $\alpha, \beta$ not constants, then assume solution of the form $\boxed{y = x^r \sum_{n=0}^{\infty} a_n x^n}$

<u>example</u> $2x^2 y'' + x y' + x^2 y = 0 \qquad (x_0 = 0)$

verify $x=0$ is rsp

~~rsp~~ $\longrightarrow y'' + \frac{1}{2x} y' + \frac{1}{2} y = 0$

rsp if $\lim_{x \to 0} x \cdot \frac{1}{2x} = \frac{1}{2}$ and $\lim_{x \to 0} x^2 \cdot \frac{1}{2} = 0$ exist

so $x=0$ is rsp

$$y = \sum_{n=0}^{\infty} a_n x^{r+n} = \underbrace{a_0 x^r}_{\text{not const. in general}} + a_1 x^{r+1} + a_2 x^{r+2} + \cdots$$

$$y' = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}$$

not losing $a_0 x^r$ to differentiation, in general

$$y'' = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1)\, x^{r+n-2}$$

Sub all into $2x^2 y'' + x y' + x^2 y = 0$

$$\sum_{n=0}^{\infty} 2a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

shift

$$\text{''} \qquad + \qquad \text{''} \qquad + \sum_{n=2}^{\infty} a_{\cancel{n+2}\; n-2}\, x^{r+n} = 0$$

$$n=0: \quad \left[2a_0 (r)(r-1) + a_0 (r)\right] x^r = 0$$

$x^r \neq 0$ , $a_0 \neq 0$ in general

$$[2(r)(r-1) + r] a_0 x^r = 0$$

$$\boxed{2(r)(r-1) + r = 0}$$

indicial eq of the corresponding Euler equation

$$2r^2 - r = 0$$

$\cancel{r(2-r)}$ $r(2r-1) = 0$

$$\boxed{r = 0, \quad r = 1/2}$$

"exponents at the singularity"

$$n=1: \underbrace{[2a_1(r+1)(r) + a_1(r+1)]}_{=0} \underbrace{x^{r+1}}_{\neq 0} = 0$$

$$a_1 \underbrace{(r+1)(2r+1)}_{\text{not zero w/ either } r \text{ from above}} = 0$$

$a_1 = 0 \longrightarrow$ <u>NOT</u> always the case

$n \geq 2$: $\quad 2a_n(r+n)(r+n-1) + a_n(r+n) + a_{n-2} = 0$

$$a_n(r+n)\underbrace{[2(r+n-1)+1]}_{(2r+2n-1)} = -a_{n-2}$$

$$\boxed{a_n = \frac{-a_{n-2}}{(r+n)(2r+2n-1)} \qquad n = 2, 3, 4, 5, \ldots}$$

depends on $r$!

recurrence relation

the two solutions $y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$ and $y_2 = x^{r_2}\sum_{n=0}^{\infty} a_n x^n$ have their own recurrence

from earlier, we found $r = 0$, $r = \frac{1}{2}$, $a_1 = 0$

$\underline{r=0}$ recurrence: $a_n = \dfrac{-a_{n-2}}{(n)(2n-1)}$ $n \geq 2$

indices differ by 2 → split into even / odd

$n = \text{even} = 2, 4, 6, \ldots$

$$a_2 = \frac{-a_0}{6} = -\frac{1}{6} a_0$$

$$a_4 = \frac{-a_2}{28} = -\frac{1}{28}\left(-\frac{1}{6} a_0\right) = \frac{1}{168} a_0$$

$$a_6 = \frac{-a_4}{66} = -\frac{1}{66}\left(\frac{1}{168} a_0\right) = -\frac{1}{11088} a_0$$

$n = \text{odd} = 3, 5, 7, 9, \ldots$

$$a_3 = \frac{-a_1}{15} = 0$$

$$a_5 = \frac{-a_3}{(-)} = 0$$

$$a_7 = 0 \ \ldots$$

first solution: $x^r \sum_{n=0}^{\infty} a_n x^n$

$$= x^0 \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right)$$

$$= a_0 - \frac{1}{6} a_0 x^2 + \frac{1}{168} a_0 x^4 - \frac{1}{11088} a_0 x^6 + \cdots$$

$$= a_0 \underbrace{\left( 1 - \frac{1}{6} x^2 + \frac{1}{168} x^4 - \frac{1}{11088} x^6 + \cdots \right)}_{y_1}$$

scaling constant

$$\boxed{y_1 = 1 - \frac{1}{6} x^2 + \frac{1}{168} x^4 - \frac{1}{11088} x^6 + \cdots}$$

repeat for next $r \rightarrow r = \frac{1}{2}$

recurrence: $a_n = \dfrac{-a_{n-2}}{(\frac{1}{2}+n)(2n)} = \dfrac{-a_{n-2}}{n+2n^2}$

second solution : $\displaystyle x^r \sum_{n=0}^{\infty} a_n x^n$

$$= x^{1/2}\left(a_0 - \frac{1}{10}a_0 x^2 + \frac{1}{360}a_0 x^4 - \dots\right)$$

$$= a_0 \underbrace{\left(x^{1/2} - \frac{1}{10}x^{5/2} + \frac{1}{360}x^{9/2} - \dots\right)}_{y_2}$$

$$\boxed{y_2 = x^{1/2} - \frac{1}{10}x^{5/2} + \frac{1}{360}x^{9/2} - \dots}$$

general solution: $y = c_1 y_1 + c_2 y_2$

depend on initial conditions

if $r_1 = r_2 \rightarrow$ ugly solutions

$|r_1 - r_2| =$ integer $\rightarrow$ VERY ugly solutions