5.6 Series Solutions Near Reg. Singular Point (part 2)

First, a quicker way to get indicial eq. of the corresponding Euler eq.

from last time: $2x^2y'' + xy' + x^2y = 0$

last time we sub in $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

we got $2(r)(r-1) + r = 0$

faster way: $y'' + \frac{\cancel{x}1}{2x} y' + \frac{1}{2} y = 0$

$x=0$ is rsp if $\lim_{x\to 0} x \cdot \frac{1}{2x}$ and $\lim_{x \to 0} x^2 \cdot \frac{1}{2}$ exist

$$\lim_{x\to 0} x \cdot \frac{1}{2x} = \frac{1}{2} = p_0$$

$$\lim_{x\to 0} x^2 \cdot \frac{1}{2} = 0 = q_0$$

the corresponding Euler ~~eq~~ eq is

$$\boxed{x^2 y'' + p_0 x y' + q_0 y = 0}$$

recall $x^2 y'' + \alpha x y' + \beta y = 0$ has

indicial eq $r(r-1) + \alpha r + \beta = 0$

here, we have $x^2 y'' + \frac{1}{2} x y' + 0 y = 0$

so indicial is $r(r-1) + \frac{1}{2} r = 0$

$2r(r-1) + r = 0$

solution: $y = x^r \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{\text{close to } a_0}$

so first-order approx. of series solution is Euler solution

example $x^2y'' + 3(\sin x)y' - 2y = 0$

sub in $y = x^r \sum_{n=0}^{\infty} a_n x^n$

need Taylor series of $\sin x$

solution takes time

$$y'' + \frac{3\sin x}{x^2} y' - \frac{2}{x^2} y = 0$$

$$p_0 = \lim_{x\to 0} x \cdot \frac{3\sin x}{x^2} = \lim_{x\to 0} \frac{3\sin x}{x} = 3$$

$$q_0 = \lim_{x\to 0} x^2 \cdot \left(-\frac{2}{x^2}\right) = -2$$

corresponding Euler: $x^2y'' + 3xy' - 2y = 0$

indicial: $r(r-1) + 3r - 2 = 0$

$$r^2 + 2r - 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$

approx. of solution near zero

Euler solution: $y = c_1 |x|^{-1+\sqrt{3}} + c_2 |x|^{-1-\sqrt{3}}$

Roots of Indicial Eq. and Solution Forms

if $r_1 \neq r_2$ and $r_1 - r_2 \neq$ integer

larger root is normally r_1

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \underbrace{a_n(r_1)}_{r=r_1 \text{ in recurrence}} x^n$$

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n$$

alternate form: sometimes we let $a_0 = 1$

(scaling constant that can be absorbed by c_1 and c_2 in $y = c_1 y_1 + c_2 y_2$)

$$y_1 = x^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right]$$

$$y_2 = x^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \right]$$

if $r_1 = r_2 = r$

y_1 is obtained the same way as above

$$y_2 = y_1 \ln x + x^r \sum_{n=1}^{\infty} b_n x^n$$

Do NOT memorize this

if $r_1 - r_2 =$ integer

y_2 obtained as in case $r_1 \neq r_2$ may or may not be linearly independent from y_1

(usually not known before solving)

$$y_2 = a\, y_1 \ln x + \underbrace{x^{r_2} \sum_{n=0}^{\infty} c_n x^n}_{\text{or } x^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n x^n\right]}$$

Do NOT memorize this

if original y_2 <u>is</u> ~~indepent f~~ independent from y_1 then $a = 0$

<u>example</u>

$$xy'' + (1-x)y' - y = 0$$

$$y'' + \frac{1-x}{x} y' - \frac{1}{x} y = 0$$

$$p_0 = \lim_{x \to 0} x \cdot \frac{1-x}{x} = 1 \qquad q_0 = \lim_{x \to 0} x^2 \left(-\frac{1}{x}\right) = 0$$

indicial eq: $r(r-1) + r + 0 = 0$

$$r^2 = 0 \rightarrow r = 0, 0$$

to get y_1, sub in $y = x^r \sum_{n=0}^{\infty} a_n x^n$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$\vdots$

$$y_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

$$y_2 = y_1 \ln x + x^{r \to 0} \sum_{n=1}^{\infty} b_n x^n$$

$$= y_1 \ln x + (b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots)$$

series

$$y_2' = \frac{y_1}{x} + y_1' \ln x + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + 5b_5 x^4 + \cdots$$

$$y_2'' = \frac{x y_1' - y_1}{x^2} + \frac{y_1'}{x} + y_1'' \ln x + 2b_2 + 6b_3 x + 12 b_4 x^2 + 20 b_5 x^3 + 30 b_6 x^4 + \cdots$$

sub into $\boxed{xy'' + (1-x)y' - y = 0}$

$2y_1'$

$$\frac{x y_1' - y_1}{x} + y_1' + \cancel{y_1'' x \ln x} + 2b_2 x + 6b_3 x^2 + 12 b_4 x^3 + 20 b_5 x^4 + \cdots$$

$$+ \frac{y_1}{x} + \cancel{y_1' \ln x} + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + 5b_5 x^4 + \cdots$$

$$- y_1 - \cancel{y_1' x \ln x} - b_1 x - 2b_2 x^2 - 3b_3 x^3 - 4b_4 x^4 - 5b_5 x^5 - \cdots$$

$$- \cancel{y_1 \ln x} - b_1 x - b_2 x^2 - b_3 x^3 - b_4 x^5 - \cdots = 0$$

now ~~sb~~ sub in $y_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$$y_1' = 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \frac{5}{5!}x^4 + \dots$$

Collect coefficients of x^n, set each to zero

x^0: $2 - 1 + b_1 = 0 \rightarrow b_1 = -1$

x^1: $2 + 2b_2 + 2b_2 - 1 - b_1 - b_1 = 0 \rightarrow b_2 = -\frac{3}{4}$

x^2: $\dots$

$$y_2 = y_1 \ln x + \underbrace{x^r \sum_{n=0}^{\infty} b_n x^n}_{x^r(b_1 + b_2 x^2 + \dots)}$$