6.1 Laplace Transform

is a type of integral transform

named after Pierre-Simon de Laplace (1749-1827)
but application to differential eqs. in ch. 6
is due to Oliver Heaviside (1850-1925)

definition of Laplace Transform

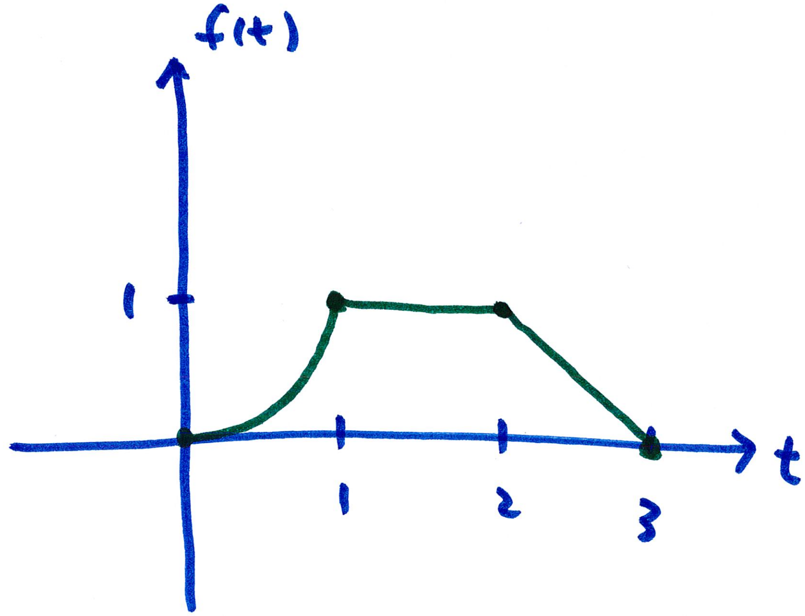
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{"kernel" of transform}} \cdot f(t)\, dt$$

almost any $f(t)$ that is at least piecewise continuous can be transformed.

finite number of discontinuities

example

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \\ 3-t & 2 < t \le 3 \end{cases}$$

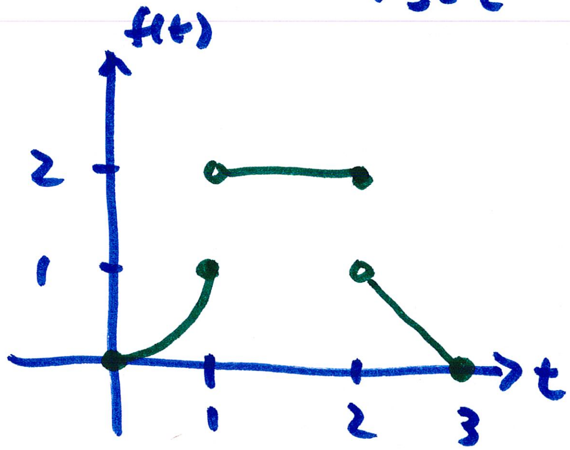


continuous

so Laplace transform exists

example

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 2 & 1 < t \le 2 \\ 3-t & 2 < t \le 3 \end{cases}$$

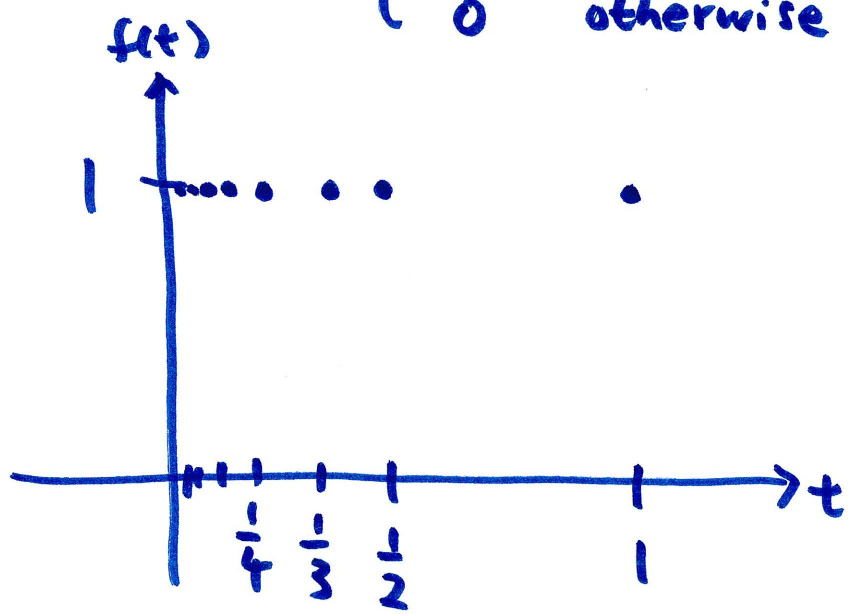


2 discontinuities

so is piecewise continuous

example

$$f(t) = \begin{cases} 1 & \text{if } t = \frac{1}{n} \text{ where } n = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$



infinitely many

discontinuities

Transform of common functions

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \lim_{a\to\infty} \int_0^a e^{-st} \, dt$$

$$= \lim_{a\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_0^a$$

$$= \lim_{a\to\infty} \left(\underbrace{-\frac{1}{s} e^{-sa}}_{\to 0 \text{ if } s>0} + \frac{1}{s}\right)$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0}$$

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st} \cdot c \, dt = c \int_0^\infty e^{-st} \, dt$$

$$= c\,\mathcal{L}\{1\} = \frac{c}{s} \quad s>0$$

LT is a <u>linear</u> process

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = \lim_{a\to\infty} \int_0^a t e^{-st}\,dt$$

by parts

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$= \lim_{a\to\infty} \left(-\frac{t}{s}e^{-st}\Big|_0^a + \frac{1}{s}\int_0^a e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} \left[\left(-\frac{a}{s}e^{-sa}\right) - \frac{1}{s^2}e^{-st}\Big|_0^a \right]$$

$$= \lim_{a\to\infty} \left(\underbrace{-\frac{a}{s}e^{-sa} - \frac{1}{s^2}e^{-sa}}_{0 \text{ if } s>0} + \frac{1}{s^2} \right)$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0}$$

$$\mathcal{L}\{1+t\} = \int_0^\infty e^{-st}(1+t)\,dt$$

$$= \int_0^\infty e^{-st}\cdot 1\,dt + \int_0^\infty e^{-st}\cdot t\,dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{t\}$$

so $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\}$

but <u>NOT</u> true for multiplication

example $\mathcal{L}\{e^{at}\cosh bt\}$ $\qquad \cosh bt = \dfrac{e^{bt}+e^{-bt}}{2}$

$$= \mathcal{L}\left\{\tfrac{1}{2}e^{at}(e^{bt}+e^{-bt})\right\}$$

$$= \mathcal{L}\left\{\tfrac{1}{2}e^{(a+b)t}+\tfrac{1}{2}e^{(a-b)t}\right\}$$

$$= \mathcal{L}\left\{\tfrac{1}{2}e^{(a+b)t}\right\}+\mathcal{L}\left\{\tfrac{1}{2}e^{(a-b)t}\right\}$$

$$= \frac{1}{2}\int_0^\infty e^{-st}e^{(a+b)t}\,dt + \frac{1}{2}\int_0^\infty e^{-st}e^{(a-b)t}\,dt$$

$$= \frac{1}{2}\lim_{c\to\infty}\int_0^c e^{(a+b-s)t}\,dt + \frac{1}{2}\lim_{c\to\infty}\int_0^c e^{(a-b-s)t}\,dt$$

$$= \frac{1}{2}\lim_{c\to\infty}\frac{1}{a+b-s}e^{(a+b-s)t}\Big|_0^c$$

$$+\frac{1}{2}\lim_{c\to\infty}\frac{1}{a-b-s}e^{(a-b-s)t}\Big|_0^c$$

$$= \frac{1}{2} \lim_{c\to\infty} \left[\frac{1}{a+b-s} e^{(a+b-s)c} - \frac{1}{a+b-s} \right]$$

$$+ \frac{1}{2} \lim_{c\to\infty} \left[\frac{1}{a-b-s} e^{(a-b-s)c} - \frac{1}{a-b-s} \right]$$

converges if $a+b-s<0$ AND $a-b-s<0$

$s-a>b$ AND $s-a>-b$

$$\boxed{\Rightarrow \quad s-a>|b|}$$

$$= \frac{1}{2} \left(\frac{-1}{a+b-s} - \frac{1}{a-b-s} \right) = \frac{1}{2} \left(\frac{1}{s-a-b} + \frac{1}{s-a+b} \right)$$

$$= \frac{1}{2} \frac{2(s-a)}{(s-a)^2-b^2} = \frac{s-a}{(s-a)^2-b^2}, \quad s-a>|b|$$