6.2 Solution of Initial-Value Problems

Solve $y'' + p(t)y' + g(t)y = f(t)$

with initial conditions $y(t_0) = y_0 \quad y'(t_0) = y_1$

LT definition $\int_0^\infty e^{-st} f(t)\,dt$

in practice, we use a Table of LT's

(to perform transforms and inverse transforms)

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\,ds$$

example If $\mathcal{L}\{f(t)\} = F(s) = \dfrac{2s-3}{s^2-4}$, find $f(t)$

find closest matches from Table

from Table: #7 $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \sinh at$

#8 $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$

$$\frac{2s-3}{s^2-2^2} = \frac{2s}{s^2-2^2} - \frac{3}{s^2-2^2}$$

want a 2 here

$$= 2\cdot\underbrace{\frac{s}{s^2-2^2}}_{\#8} - 3\cdot\underbrace{\frac{1}{s^2-2^2}}_{\text{almost }\#7}$$

$$= 2\cdot\frac{s}{s^2-2^2} - \frac{3}{2}\cdot\frac{2}{s^2-2^2}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-2^2}\right\} = \boxed{2\cosh 2t - \frac{3}{2}\sinh 2t}$$

example $F(s) = \dfrac{3s}{s^2 - s - 6} = \dfrac{3s}{(s+2)(s-3)}$

closest match: #2 $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

we can do partial fraction expansion

$$\frac{3s}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3}$$

$$3s = (s+2)(s-3)\frac{A}{s+2} + (s+2)(s-3)\frac{B}{s-3}$$

$$3s = A(s-3) + B(s+2)$$

$$3s + 0 = (A+B)s + (-3A+2B)$$

$$A+B = 3 \qquad A = 3 - \frac{9}{5} = \frac{6}{5}$$

$$\left.\begin{aligned} -3A+2B &= 0 \\ 3A+3B &= 9 \end{aligned}\right\} \text{add: } 5B = 9$$

$$B = \frac{9}{5}$$

$$\frac{3s}{(s+2)(s-3)} = \frac{6}{5}\cdot\frac{1}{s+2} + \frac{9}{5}\cdot\frac{1}{s-3}$$

$$\boxed{f(t) = \frac{6}{5}e^{-2t} + \frac{9}{5}e^{3t}}$$

LT of derivatives of $y(t)$ → usually unknown

$$\mathcal{L}\{y\} = Y$$

$$\mathcal{L}\{y'\} = \int_0^\infty e^{-st}\cdot y'\,dt = \lim_{a\to\infty}\int_0^a e^{-st}\cdot y'\,dt$$

by parts: $u = e^{-st}$ $dv = y'dt$

$du = -se^{-st}dt$ $v = y$

$$= \lim_{a\to\infty}\left(y(t)e^{-st}\Big|_0^a + s\int_0^a e^{-st}\,y(t)\,dt\right)$$

$$= \lim_{a\to\infty} \left(\underset{s>0}{\cancelto{0}{y(a)e^{-sa}}} - y(0) \right) + s \underbrace{\int_0^\infty e^{-st}\, y(t)\,dt}_{\mathcal{L}\{y\} = Y}$$

so $\boxed{\mathcal{L}\{y'\} = sY - y(0)}$

Similarly,

$$\mathcal{L}\{y''\} = s^2 Y - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y'''\} = s^3 Y - s^2 y(0) - s\,y'(0) - y''(0)$$

$$\mathcal{L}\{y^{(4)}\} = s^4 Y - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)$$

<u>example</u> Solve $y'' - 2y' + 2y = 0$ $\quad y(0) = 0, \quad y'(0) = 1$

LT both sides

$$s^2Y - sy(0) - y'(0) - 2[sY - y(0)] + 2Y = 0$$

Sub in IC's

$$s^2Y - 1 - 2sY + 2Y = 0$$

$$\underbrace{(s^2 - 2s + 2)}_{\text{characterist equation (roots } \to e^{at})} Y = 1$$

$$Y = \underbrace{\frac{1}{s^2 - 2s + 2}} \qquad \mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2}$$

can't be factored nicely

$\Rightarrow$ complete the square

because #9: $\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{at} \sin bt$

#9: $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + b^2}\right\} = e^{at} \cos bt$

$$Y = \frac{1}{s^2-2s+2} = \frac{1}{(s^2-2s+1)+1} = \frac{1}{(s-1)^2+1^2}$$

so $\boxed{y(t) = e^t \sin t}$

LT is very useful when right side is discontinuous

<u>example</u>

$$y'' + y = \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t < \infty \end{cases} \qquad y(0)=1,\ y'(0)=0$$

LT both sides:

$$\mathcal{L}\{f(t)\} \quad \text{where} \quad f(t) = \begin{cases} 1 & 0 \le t < \pi \\ 0 & \pi \le t < \infty \end{cases}$$

$$= \int_0^\infty e^{-st} \cdot f(t)\, dt = \int_0^\pi e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\pi$$

$$= -\frac{1}{s} e^{-s\pi} + \frac{1}{s} = \frac{1-e^{-s\pi}}{s}$$

LT of left side:

$$s^2Y - sy(0) - y'(0) + Y = (s^2+1)Y - s$$

$$(s^2+1)Y - s = \frac{1-e^{-s\pi}}{s}$$

$$(s^2+1)Y = s + \frac{1-e^{-s\pi}}{s}$$

unit step function (6.3)

$$Y = \underbrace{\frac{s}{s^2+1}}_{\cos t} + \underbrace{\frac{1-e^{-s\pi}}{s(s^2+1)}}_{\text{due to the discontinuous right side}}$$