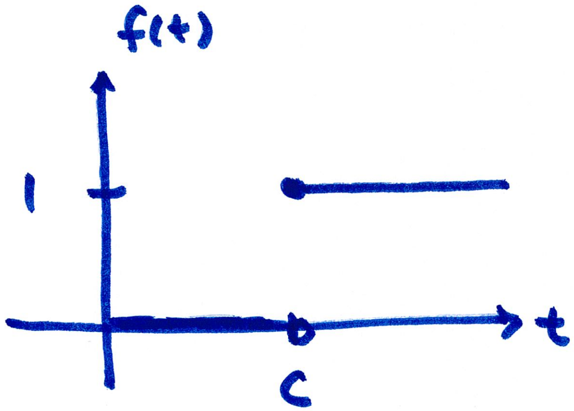
6.3 Step Functions

define unit step function $u_c(t) = \begin{cases} 0 & \text{if } 0 \le t < c \\ 1 & \text{if } t \ge c \end{cases}$



piecewise continuous so LT exists.

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} \cdot u_c(t)\, dt$$

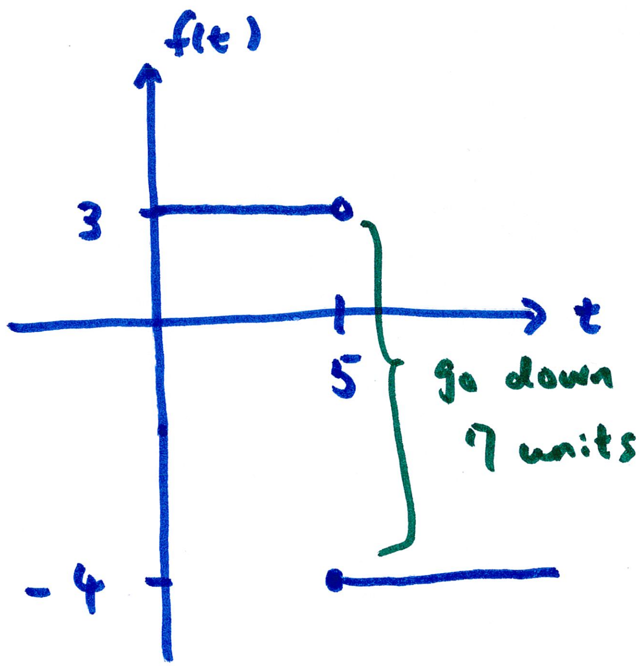
$$= \int_c^\infty e^{-st}\, dt = \lim_{a \to \infty} \left(-\frac{1}{s} e^{-st} \Big|_c^a \right)$$

$$= \lim_{a \to \infty} \left(\cancelto{0}{-\frac{1}{s} e^{-sa}} + \frac{1}{s} e^{-cs} \right) = \frac{1}{s} e^{-cs}$$

if $\underline{\underline{s > 0}}$

$$\boxed{\mathcal{L}\{u_c(t)\} = e^{-cs} \cdot \frac{1}{s}}$$

<u>example</u> Find LT of $f(t) = \begin{cases} 3 & 0 \le t < 5 \\ -4 & 5 \le t < \infty \end{cases}$



rewrite $f(t)$ in terms of $u_c(t)$

$$f(t) = 3 + u_5(t) \cdot (-7)$$

$$= 3 - 7u_5(t)$$

$$F(s) = \mathcal{L}\{f(t)\} = \frac{3}{s} - 7 \cdot e^{-5s} \cdot \frac{1}{s} = \frac{3 - 7e^{-5s}}{s}$$

example

$$f(t) = \begin{cases} 3 & 0 \le t < 1 \\ -1 & 1 \le t < 3 \\ 1 & 3 \le t < \infty \end{cases}$$

$$f(t) = 3 - 4u_1(t) + 2u_3(t)$$

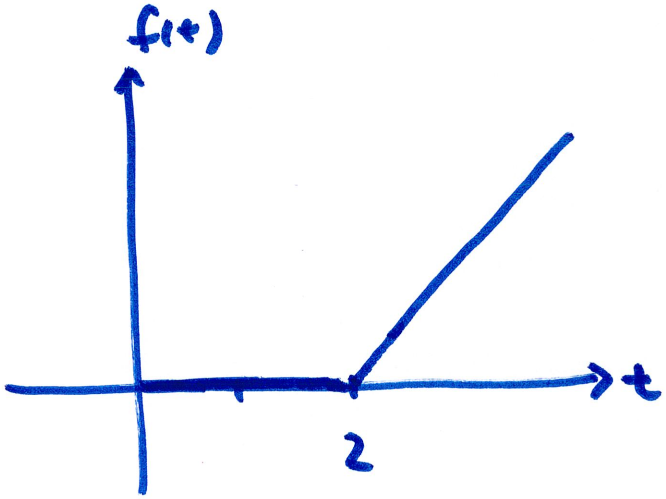
$$F(s) = \frac{3}{s} - 4e^{-s} \cdot \frac{1}{s} + 2e^{-3s} \cdot \frac{1}{s}$$

$$\mathcal{L}\{u_c(t) \cdot k\} = e^{-cs} \cdot \frac{k}{s}$$

if not a constant, transform is more complicated

$$\mathcal{L}\{u_c(t) \cdot f(t)\} \neq e^{-cs} \mathcal{L}\{f(t)\}$$

<u>example</u> $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{cases}$



$f(t) = u_2(t) \cdot (t-2)$

$F(s) \neq e^{-2s} \mathcal{L}\{t-2\}$

use definition first:

$$F(s) = \int_0^\infty e^{-st} u_2(t)(t-2)\,dt = \int_2^\infty e^{-st}(t-2)\,dt$$

$u = t-2 \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$$F(s) = \cancelto{0}{-\frac{t-2}{s}e^{-st}}\Big|_2^{\infty} + \frac{1}{s}\int_2^{\infty} e^{-st}\,dt$$

$s>0$

$$= -\frac{1}{s^2}e^{-st}\Big|_2^{\infty} = 0 - -e^{-2s}\frac{1}{s^2} = e^{-2s}\cdot\underbrace{\frac{1}{s^2}}_{\mathcal{L}\{t\}}$$

so $\mathcal{L}\{u_2(t)\cdot(t-2)\}$ is NOT $e^{-2s}\mathcal{L}\{t-2\}$

but is $e^{-2s}\mathcal{L}\{t-2 \text{ translated LEFT by } 2\}$

$$\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-cs}\underbrace{\mathcal{L}\{f(t)\}}$$

transform function activated by $u_c(t)$ c units <u>LEFT</u>

example

$$f(t) = \begin{cases} 1 & 0 \le t < 2 \\ e^{-(t-3)} & t \ge 2 \end{cases}$$

$$= 1 + u_2(t) \cdot \left[e^{-(t-3)} - 1 \right]$$

$$\mathcal{L}\{u_c(t) \cdot f(t-c)\} = e^{-cs}\, \mathcal{L}\{f(t)\}$$

$$F(s) = \frac{1}{s} + e^{-2s}\, \mathcal{L}\Big\{ \underbrace{e^{-(t+2-3)} - 1}_{e^{-(t-3)} - 1 \text{ shifted LEFT by } 2} \Big\}$$

$2 \to u_2(t)$

$$= \frac{1}{s} + e^{-2s}\, \mathcal{L}\{e^{-t+1} - 1\}$$

$$= \frac{1}{s} + e^{-2s}\, \mathcal{L}\{e \cdot e^{-t} - 1\}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= \boxed{\frac{1}{s} + e^{-2s}\left(e \cdot \frac{1}{s+1} - \frac{1}{s}\right)}$$

inverse transform is the reverse of the above

$$\mathcal{L}\{u_c(t) \cdot f(t-c)\} = e^{-cs} F(s)$$

<u>example</u>

$$F(s) = \frac{2e^{-\frac{\pi}{4}s}}{s^2+4} \qquad f(t) = ?$$

$$= e^{-\frac{\pi}{4}s} \cdot \boxed{\frac{2}{s^2+4}}$$

LT of the function activated by $u_c(t)$ AFTER shifting LEFT by c

$\longrightarrow \sin 2t$

$$f(t) = u_{\frac{\pi}{4}}(t) \cdot \sin 2(t \underbrace{- \tfrac{\pi}{4}})$$

shift $\sin 2t$ RIGHT $\frac{\pi}{4}$ units example $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2-2s+2} = e^{-2s} \cdot \dfrac{2(s-1)}{s^2-2s+2}$

$$= 2e^{-2s} \cdot \frac{s-1}{s^2-2s+2}$$

$$= 2e^{-2s} \cdot \boxed{\frac{s-1}{(s-1)^2+1}}$$

LT of function activated by $u_2(t)$ AFTER shifting LEFT by 2

turns into

$e^t \cos(t)$ → shift RIGHT by 2

~~f(t)~~

$$f(t) = 2u_2(t) \cdot e^{(t-2)} \cos(t-2)$$