Exam 1: RAWL 1062 8:40 – 9:40

11 questions (7 multiple choice)

Review

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| < 1$

Series solutions near ordinary pt

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad P(x_0) \neq 0$$

solution form: $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

generally $a_0 \neq 0$

$y(x_0) = a_0$

$y'(x_0) = a_1$

~~$y^{(n)}(x_0)$~~ $\boxed{y^{(n)}(x_0) = n!\, a_n}$

example $y'' + 4y = 0 \qquad x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$n \geq 0$: $a_{n+2}(n+2)(n+1) + 4a_n = 0$

$$a_{n+2} = \frac{-4a_n}{(n+2)(n+1)} \qquad n = 0, 1, 2, 3, \ldots$$

<u>recurrence relation</u>

<u>Singular Points</u>

$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad P(x_0) = 0$

<u>regular</u> if $\lim\limits_{x \to x_0} (x - x_0)\dfrac{Q(x)}{P(x)}$ and $\lim\limits_{x \to x_0} (x - x_0)^2 \dfrac{R(x)}{P(x)}$

both are <u>finite</u>

if $p_0 = \lim\limits_{x \to x_0} (x - x_0)\dfrac{Q(x)}{P(x)} \qquad q_0 = \lim\limits_{x \to x_0} (x - x_0)^2 \dfrac{R(x)}{P(x)}$

then corresponding Euler eq: $x^2 y'' + p_0 x y' + q_0 y = 0$

$$\boxed{\text{indicial eq: } \quad r(r-1)+p_0 r+q_0=0}$$

Sub solution form $y=x^r\sum_{n=0}^{\infty} a_n (x-x_0)^n$ into DE.

example $\quad x^2y''+xy'-(1-x)y=0$

$$y''+\frac{1}{x}y'-\left(\frac{1-x}{x^2}\right)y=0 \qquad \text{note } x=0 \text{ is singular}$$

$$p_0=\lim_{x\to 0} x\cdot\frac{1}{x}=1 \qquad q_0=\lim_{x\to 0} x^2\cdot -\frac{(1-x)}{x^2}=-1$$

indicial: $\quad r(r-1)+r-1=0$

$$r^2-1=0 \qquad r=\pm 1$$

y_1: from <u>larger</u> root

$r=1$ $\qquad y = \sum_{n=0}^{\infty} a_n x^{r+n} = \sum_{n=0}^{\infty} a_n x^{1+n}$

$$y' = \sum_{n=0}^{\infty} a_n (1+n) x^{n}$$

$$y'' = \sum_{n=0}^{\infty} a_n (1+n)(n) x^{n-1}$$

$$\sum_{n=0}^{\infty} a_n (1+n)(n) x^{n+1} + \sum_{n=0}^{\infty} a_n (1+n) x^{n+1}$$

$$- \sum_{n=0}^{\infty} a_n x^{1+n} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+2}} = 0$$

$$(\cdot\cdot) + (\cdots) + (\cdots) + \sum_{n=1}^{\infty} a_{n-1} x^{n+1} = 0$$

$n=0$: $\quad a_0 - a_0 = 0$

$n \geq 1$: $\quad (n)(1+n) a_n + (1+n) a_n - a_n + a_{n-1} = 0$

$$a_n\left[(n)(1+n)+(1+n)-1\right] = -a_{n-1}$$

$$a_n = \frac{-a_{n-1}}{(n)(n+1)+n} \qquad n=1,2,3,4,\ldots$$

Euler eq solutions

$$x^2y''+\alpha x y'+\beta y=0 \qquad \alpha,\beta \text{ constants}$$

Solution is <u>NOT</u> a series!

indicial: $r(r-1)+\alpha r+\beta=0$

$r_1 \neq r_2$: $\quad y = c_1|x|^{r_1}+c_2|x|^{r_2}$

$r_1 = r_2$: $\quad y = c_1|x|^{r}+c_2|x|^{r}\ln|x|$

$r=\lambda \pm i\mu$: $\quad y = c_1|x|^{\lambda}\cos(\mu\ln|x|)+c_2|x|^{\lambda}\sin(\mu\ln|x|)$

Lower Bound of Radius of Convergence

$P(x)y'' + Q(x)y' + R(x)y = 0$

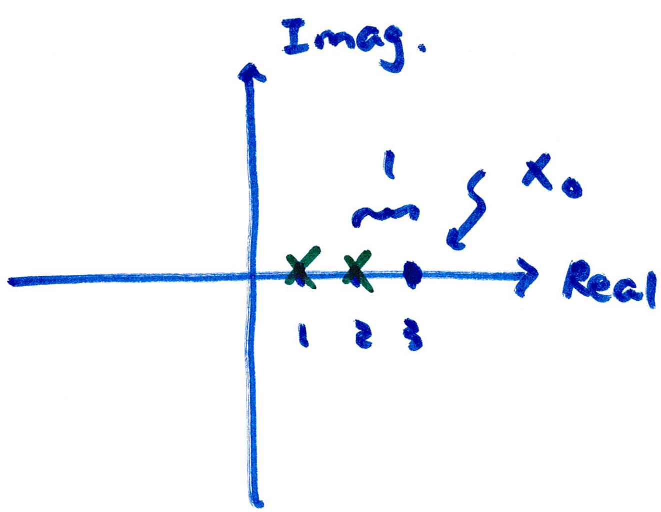
find poles: $P(x) = 0$

then find distance from x_0 to the closest pole

example $(1-x)(2-x)y'' + 3(x-1)y' + 6y = 0$

$x_0 = 3$

poles: $x=2$, $x=1$



so rad. of conv. is at least 1.

Laplace Transform

Table attached to exam

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$$

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\} = sY - y(0)$$

$$\mathcal{L}\{y^{(4)}\} = s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$$

$$\mathcal{L}\{u_c(t) \cdot \underbrace{f(t-c)}\} = e^{-cs}\,\mathcal{L}\{f(t)\}$$

shift LEFT c units

(change t to t+c)

example

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 2t & 1 \le t < 2 \\ 3 & t \ge 2 \end{cases}$$

in terms of unit step:

$$f(t) = 1 + u_1(t) \cdot (2t - 1) + u_2(t) \cdot (3 - 2t)$$

$$F(s) = \frac{1}{s} + e^{-s} \mathcal{L}\{\underbrace{2(t+1) - 1}_{2t+1}\} + e^{-2s} \mathcal{L}\{\underbrace{3 - 2(t+2)}_{-2t-1}\}$$

$$F(s) = \frac{1}{s} + e^{-s}\left(\frac{2}{s^2} + \frac{1}{s}\right) + e^{-2s}\left(-\frac{2}{s^2} - \frac{1}{s}\right)$$

<u>example</u> $F(s) = \dfrac{e^{-\pi s}}{(s^2+1)s^2} = e^{-\pi s} \cdot \dfrac{1}{s^2(s^2+1)}$

$$\frac{1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = (As+B)(s^2+1) + (Cs+D)(s^2)$$

$$= As^3 + Bs^2 + As + B + Cs^3 + Ds^2$$

$$1 = (A+C)s^3 + (B+D)s^2 + As + B$$

$$A=0, \quad B=1, \quad C=0, \quad D=-1$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$$

$$f(t) = u_\pi(t) \cdot \left[(t-\pi) - \sin(t-\pi)\right]$$