Exam 2

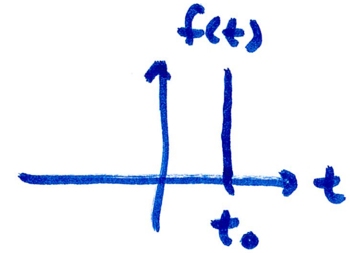
in **GRIS 103**

11 questions, 6 multiple choice

Impulse function

$$\delta(t-t_0) = 0 \quad \text{if } t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$



$$\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$$

inverse LT produces step function

$$y'' - y = \delta(t-1) \qquad y(0) = y'(0) = 0$$

$$s^2 Y - Y = e^{-s}$$

$$(s^2-1)Y = e^{-s}$$

$$Y = e^{-s}\frac{1}{s^2-1}$$

$$y(t) = u_1(t)\cdot \sinh(t-1)$$

Convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$\mathcal{L}\{f(t) * g(t)\} = F(s)\, G(s)$$

$$\mathcal{L}\left\{ \int_0^t \underbrace{\sin(t-\tau)}_{f(t-\tau)}\, \underbrace{\cos\tau}_{g(\tau)}\, d\tau \right\}$$

$$= \mathcal{L}\{\sin(t)\}\, \mathcal{L}\{\cos(t)\}$$

$$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

$$F(s) = \frac{s}{(s+1)(s^2+4)} = \underbrace{\frac{1}{s+1}}_{e^{-t}} \cdot \underbrace{\frac{s}{s^2+4}}_{\frac{1}{2}\cos 2t}$$

$$= \int_0^t e^{-(t-\tau)} \cdot \frac{1}{2} \cos 2\tau\, d\tau$$

Homogeneous Systems $\vec{x}' = A\vec{x}$ (up to 3x3)

if eigenvalues are distinct $\quad \vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

complex $\quad$ find one solution, separate into real / imaginary parts, use them to form general solution.

repeated $\quad$ if A is complete, just like distinct case.

if A is defective, need generalized eigenvector $\vec{u}$ such that $(A-\lambda I)\vec{u} = \vec{v}$

$$\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}\left[\vec{v}t + \vec{u}\right]$$

phase portrait

distinct e-values : origin is source or sink if λ's have same sign
saddle point if mixed signs

complex : stable/unstable spirals

real part < 0 ; real part > 0

repeated : "improper node", similar to distinct case.

Fundamental Matrices

$$\vec{x}\,' = A\vec{x}$$

$$\vec{x} = c_1 \underbrace{e^{\lambda_1 t}\vec{v}_1}_{\vec{x}_1} + c_2 \underbrace{e^{\lambda_2 t}\vec{v}_2}_{\vec{x}_2}$$

$$\Psi(t) = \left[\vec{x}_1 \;\; \vec{x}_2\right]$$

fundamental solutions as columns.

$$\vec{x} = \Psi\vec{c}$$

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_0) \cancel{X(t_0)}$$

$$\vec{x} = \cancel{\Phi(t)\vec{x}(t_0)} \quad \Phi(t)\vec{x}(t_0)$$

Nonhomogeneous

$$\vec{x}\,' = A\vec{x} + \vec{g}$$

undetermined coefficients

variation of parameters

$$\vec{x} = \Psi\vec{u} \qquad \Psi\vec{u}\,' = \vec{g}$$

Euler method

$$y' = f(t, y)$$

no improved Euler

$$y_{n+1} = y_n + f(t_n, y_n)h$$

Boundary Value Problem

$$y'' + \lambda y = 0$$

eigenvalue: λ such y is a nontrivial solution

eigenfunction: solution is linear combination of these

$y'' + \lambda y = 0 \qquad y'(0) = y'(3\pi) = 0$

find eigenvalues and eigenfunctions

<u>$\lambda > 0$</u>

$\lambda = \mu^2$

$r^2 + \mu^2 = 0 \qquad r = \pm i\mu$

$y = c_1 \cos\mu x + c_2 \sin\mu x$

$y' = -\mu c_1 \sin\mu x + \mu c_2 \cos\mu x$

$0 = \mu c_2 \longrightarrow c_2 = 0 \quad (\mu^2 = \lambda > 0 \text{ so } \mu \neq 0)$

$0 = -\mu c_1 \sin 3\pi\mu = 0 \qquad c_1 \neq 0$ (otherwise y is trivial)

$\sin 3\pi\mu = 0 \longrightarrow 3\pi\mu = n\pi \qquad n = 1, 2, 3, \ldots$

$\mu = \frac{n}{3}$

eigenvalues $\longrightarrow \lambda = \left(\frac{n}{3}\right)^2 \qquad n = 1, 2, 3$

$y = c_1 \cos\left(\frac{n}{3}x\right)$ eigenfunctions: $\cos\left(\frac{nx}{3}\right)$

<u>$\lambda = 0$</u>

$y'' = 0 \qquad y'(0) = y'(3\pi) = 0$

$y = c_1 x + c_2$

$y' = c_1 \qquad\qquad 0 = c_1$

$0 = c_1$

$y = c_2 \cdot (1)$ eigenvalue: 0

eigenfuction: 1

<u>$\lambda < 0$</u>

$y'' + \lambda y = 0 \qquad y'(0) = y'(3\pi) = 0$

$\lambda = -\mu^2$

$r^2 - \mu^2 = 0 \qquad r = \pm\mu$

$y = c_1 e^{-\mu x} + c_2 e^{\mu x}$

$y' = -\mu c_1 e^{-\mu x} + \mu c_2 e^{\mu x}$

$$0 = -\mu c_1 + \mu c_2 \rightarrow c_1 = c_2$$

$$0 = -\mu c_1 e^{-3\pi\mu} + \mu c_2 e^{3\pi\mu}$$

$$0 = -c_1 e^{-3\pi\mu} + c_1 e^{3\pi\mu} \rightarrow c_1 = 0$$

$$c_2 = 0$$

no negative eigenvalues

$$\vec{x}\,' = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 45t \\ e^{3t} \end{bmatrix}$$

$$\vec{x}_h = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

not good w/ undet. coeff.
because $\vec{g}$ has part of $\vec{x}_h$

use variation of parameters

$$\Psi = \begin{bmatrix} e^{3t} & 2e^{5t} \\ e^{3t} & e^{5t} \end{bmatrix} \qquad \vec{x} = \Psi \vec{u} \qquad \Psi \vec{u}\,' = \vec{g}$$

$$\left[\begin{array}{cc|c} e^{3t} & 2e^{5t} & 45t \\ e^{3t} & e^{5t} & e^{3t} \end{array}\right]$$

$$\left[\begin{array}{cc|c} e^{3t} & 2e^{5t} & 45t \\ 0 & -e^{5t} & e^{3t} - 45t \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2e^{2t} & 45te^{-3t} \\ 0 & 1 & -e^{-2t}+45te^{-5t} \end{array}\right]$$

$$u_2' = -e^{-2t} + 45\underbrace{t\,e^{-5t}}_{\text{by parts}}$$

$u = t \quad dv = e^{-5t}dt$

$du = dt \quad v = -\frac{1}{5}e^{-5t}$

$$u_2 = \frac{1}{2}e^{-2t} + 45\left(-\frac{t}{5}e^{-5t} + \frac{1}{5}\int e^{-5t}\,dt\right) \underbrace{+\,c_2}_{\substack{c_1,c_2=0\\ \text{gives}\\ \text{particular}\\ \text{solution}}}$$

$$u_2 = \frac{1}{2}e^{-2t} - 9t\,e^{-5t} - \frac{9}{5}e^{-5t}$$

$$u_1' = -2e^{2t}u_2' + 45te^{-3t}$$

$$= 2 - 90\underbrace{t\,e^{-3t}}_{\text{by parts}} + 45\underbrace{te^{-3t}}_{\text{by parts}}$$

$$= 2 - 45\underbrace{t\,e^{-3t}}_{\text{by parts}}$$