# Exam 1 Review

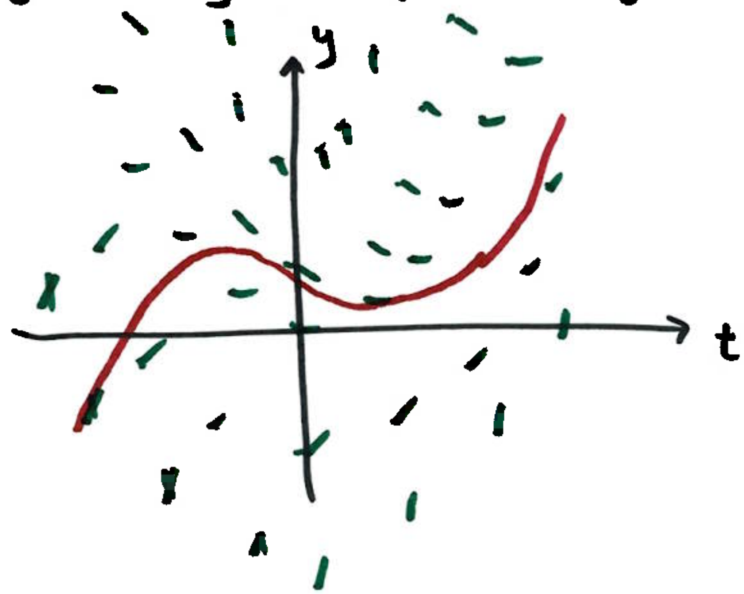
Direction field: $y' = f(t,y)$

pick $(t,y)$, graph $y'$

if no $t$ in $f(t,y)$, same slope across

if it contains $t$, best to find out where $y'=0$
then above and below

$y' = t^2 - y$ $\quad$ $y'=0 \rightarrow y = t^2$




$y > t^2$ (above)

$y' < 0$

$y < t^2$ (below)

$y' > 0$

# Types of Diff. Eqs.

functions of $t$ or constant

**1st-order linear** : $y' + p(t)\,y = g(t)$

Solution : integrating factor $\mu = e^{\int p(t)\,dt}$

multiply both sides of diff. eq.

left $\rightarrow \frac{d}{dt}(\mu y)$

$\frac{d}{dt}(\mu y) = \mu g$ then integrate and solve

example : $t y' + 4y = 12t^2$

$$y' + \frac{4}{t} y = 12t$$

$p(t)$

$$\mu = e^{\int \frac{4}{t} dt} = e^{4\ln t} = t^4$$

$$\frac{d}{dt}(t^4 y) = 12t^5$$

$$t^4 y = 2t^6 + C$$

$$y = 2t^2 + \frac{C}{t^4}$$

## Separable

$$\frac{dy}{dx} = M(x)N(y)$$

solution: $\frac{1}{N(y)}dy = M(x)\,dx$

integrate and solve

example: $\frac{dy}{dx} = xy + x = x(y+1)$

$$\frac{1}{y+1}dy = x\,dx$$

$$\ln|y+1| = \frac{1}{2}x^2 + c$$

$$|y+1| = e^c \cdot e^{\frac{1}{2}x^2}$$

$$y+1 = Ce^{\frac{1}{2}x^2}$$

$$y = Ce^{\frac{1}{2}x^2} - 1$$

homogeneous

$$\frac{dy}{dx} = f(x,y) = g\left(\frac{y}{x}\right)$$

solution: make sub $v = \frac{y}{x}$

rewrite eq. as one involving $v$ and $x$

(could be separable or linear or others)

Solve for $v$, then $y$

example: $$\frac{dy}{dx} = \frac{3x-y}{x+y}$$

$$= \frac{3-\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)} = g\left(\frac{y}{x}\right)$$

$v = \frac{y}{x}$ $\qquad y = vx$

$$\frac{dy}{dx} = v + v'x$$

$$v + v'x = \frac{3-v}{1+v}$$ no $y$!

$$x v' = \frac{3-v}{1+v} - \frac{v+v^2}{1+v}$$

$$x v' = \frac{3-2v-v^2}{1+v} \qquad \text{separable}$$

$$\frac{1+v}{v^2+2v-3}\, dv = -\frac{1}{x}\, dx$$

integrate, find v, then y.

$$u = v^2+2v-3$$

$$du = (2v+2)\,dv$$

$$= 2(v+1)\,dv$$

$$\frac{1}{2}\int \frac{1}{u}\, du = -\int \frac{1}{x}\, dx$$

$$\vdots$$

exact $\quad M(x,y) + N(x,y)\,y' = 0$

or $\quad M(x,y)\,dx + N(x,y)\,dy = 0$

such that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (or $M_y = N_x$)

Solution: $\psi(x,y) = C$

where $\dfrac{\partial \psi}{\partial x} = M$ and $\dfrac{\partial \psi}{\partial y} = N$

example: $\underbrace{(e^x \sin y - 2y \sin x - 1)}_{M}dx + \underbrace{(e^x \cos y + 2\cos x + 1)}_{N}dy = 0$

$M_y = e^x \cos y - 2\sin x$

$N_x = e^x \cos y - 2\sin x$

so exact

$\psi_x = e^x \sin y - 2y \sin x - 1$

$\psi_y = e^x \cos y + 2\cos x + 1$

pick one to integrate

$\Psi_y = e^x \cos y + 2\cos x + 1$

$\Psi = \int (e^x \cos y + 2\cos x + 1)\,dy$ → x is constant

$= e^x \sin y + 2y\cos x + y + h(x)$

$\Psi_x = e^x \sin y - 2y \sin x + h'(x) = M = e^x \sin y - 2y\sin x - 1$

so, $h'(x) = -1 \rightarrow h(x) = -x$

$\Psi(x,y) = e^x \sin y + 2y\cos x + y - x$

solution: $\Psi = C$

No exact integrating factor on exam

# Equilibrium and Stability

$y' = f(t, y)$

autonomous : $y' = f(y)$

equilibrium/critical pt : $f(y) = 0$
$(y' = 0)$

stable : solutions nearby converge onto it

unstable : " " run away from it

semi-stable : some converge onto others run away

example : $y' = y^2(y-1)(y-2)$

$y' = 0 \rightarrow y = 0,\ y = 1,\ y = 2$

$y'$   $+$   $0$   $+$   $0$   $-$   $0$   $+$


semi-stable    stable    unstable

## Existence and Uniqueness

1st-order linear

$y' + p(t)y = g(t) \quad y(t_0) = y_0$

$p, g$ continuous and contain $t_0$

~~2nd-order~~

1st-order nonlinear

$y' = f(t, y)$

$f$ and $\frac{\partial f}{\partial y}$ are continuous containing $(t_0, y_0)$

## Euler's method

$y' = f(t, y) \quad y(t_0) = y_0$

decide $h$ (step size)

$t_0$

$y_0$

$t_1 = t_0 + h$

$y_1 = y_0 + f(t_0, y_0)h$

$\vdots$

$y_n = y_{n-1} + f(t_{n-1}, y_{n-1})h$

## 2nd-order homogeneous constant-coefficient

$$ay'' + by' + cy = 0$$

characteristic eq. $ar^2 + br + c = 0$

on exam: only distinct real roots
(no complex, no repeated)

solutions: $y_1 = e^{r_1 t}$

$y_2 = e^{r_2 t}$

general solution: $y = c_1 y_1 + c_2 y_2$

no Wronskian on the exam