Laplace Transform

Initial-value problems

$$y'' + 4y' + 13y = 5e^{-2t} \qquad y(0)=0,\ y'(0)=6$$

transform both sides

$$s^2Y - s\cancel{y(0)} - y'(0) + 4(sY - \cancel{y(0)}) + 13Y = \frac{5}{s+2}$$

$y(0)=0$

$$(s^2+4s+13)Y = 6 + \frac{5}{s+2}$$

$$Y = \frac{6}{s^2+4s+13} + \underbrace{\frac{5}{(s+2)(s^2+4s+13)}}$$

$$\frac{5}{(s+2)(s^2+4s+13)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4s+13}$$

$$5 = A(s^2+4s+13) + (Bs+C)(s+2)$$

$$5 = (A+B)s^2 + (4A+2B+C)s + (13A+2C)$$

$A+B=0 \rightarrow B=-A$

$4A+2B+C=0 \rightarrow 2A+C=0 \rightarrow C=-2A$

$13A+2C=5 \rightarrow \dots \rightarrow A=\frac{5}{9},\ B=-\frac{5}{9},\ C=-\frac{10}{9}$

$$Y = \frac{6}{s^2+4s+13} + \frac{5}{9}\cdot\frac{1}{s+2} - \frac{5}{9}\cdot\frac{s+2}{s^2+4s+13}$$

complete the square

$$s^2+4s+4+9 = (s+2)^2+3^2$$

$$= \frac{6}{(s+2)^2+3^2} + \frac{5}{9}\cdot\frac{1}{s+2} - \frac{5}{9}\,\frac{s+2}{(s+2)^2+3^2}$$

$$y(t) = 2e^{-2t}\sin(3t) + \frac{5}{9}e^{-2t} - \frac{5}{9}e^{-2t}\cos(3t)$$

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step function: $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

$$f(t) = \begin{cases} 2t & 0 \leq t < \pi \\ 0 & t \geq \pi \end{cases} = 2t + u_\pi(t)(-2t) = 2t - 2u_\pi(t)(t)$$

$$F(s) = \mathcal{L}\{2t\} - 2\mathcal{L}\{u_\pi(t)\cdot t\}$$

shift LEFT by π $\rightarrow$ change t to $t+\pi$

$$= \frac{2}{s^2} - 2e^{-\pi s}\mathcal{L}\{t+\pi\} = \frac{2}{s^2} - 2e^{-\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)$$

shift RIGHT come back to t

convolution: $\mathcal{L}\left\{ \int_0^t f(\tau)\, g(t-\tau)\, d\tau \right\} = F(s)G(s)$

$$\frac{1}{s(s^2+4)} = \underbrace{\frac{1}{s}}_{F} \cdot \underbrace{\frac{1}{s^2+4}}_{G}$$

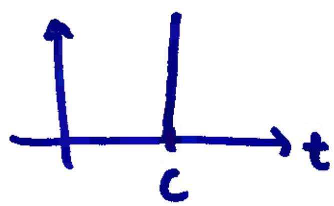
$f(t) = 1$

$g(t) = \frac{1}{2}\sin(2t)$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s(s^2+4)} \right\} = \int_0^t 1 \cdot \frac{1}{2}\sin(2(t-\tau))\, d\tau$$

$$= \int_0^t 1 \cdot \frac{1}{2}\sin(2\tau)\, d\tau$$

impulse: $\delta(t-c)$



$$\int_{-\infty}^{\infty} \delta(t-c)\, dt = 1$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

$$y'' + y = \delta(t-1) \qquad y(0) = y'(0) = 0$$

$$s^2 Y - s y(0) - y'(0) + Y = e^{-s}$$

$$(s^2+1)Y = e^{-s}$$

$$Y = e^{-s} \cdot \frac{1}{s^2+1}$$

$\to \sin(t)$

Shift RIGHT by 1 : $t \to t-1$

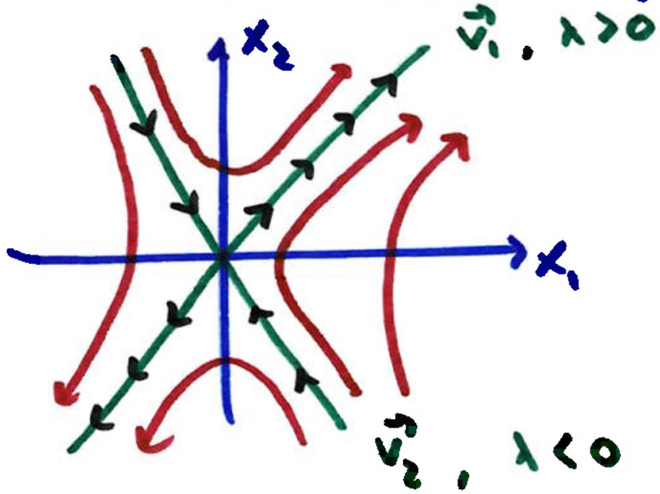
$$y = u_1(t) \sin(t-1)$$

NOT δ

$\vec{x}' = A\vec{x}$

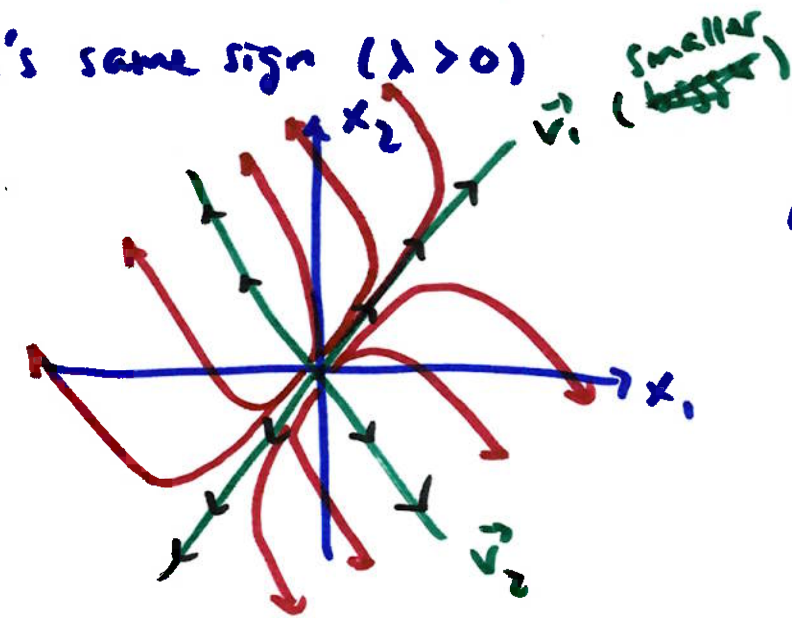
Solution: $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_1$ $\lambda, \vec{v} \rightarrow$ eigenvalues/eigenvector pairs

if λ's are real and opposite signs



origin is a saddle point

λ's same sign ($\lambda > 0$)



if $\lambda < 0$, reverse the direction

origin is a nodal source ($\lambda > 0$)

or nodal sink ($\lambda < 0$)

if λ is repeated but enough eigenvectors, nothing special

→ see two straight line solutions

if not enough, then $\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})$

one straight line solution visible on graph

generalized eigenvector

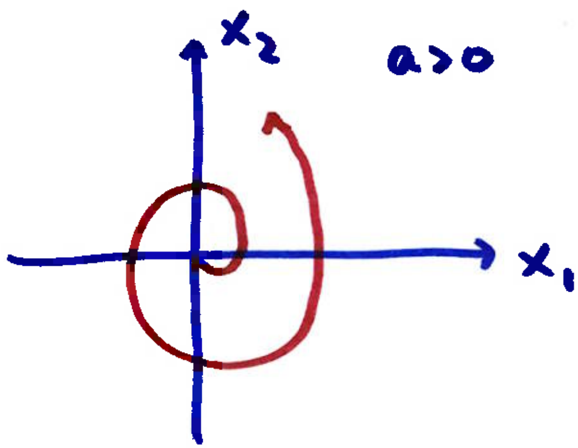
$(A - \lambda I)\vec{u} = \vec{v}$

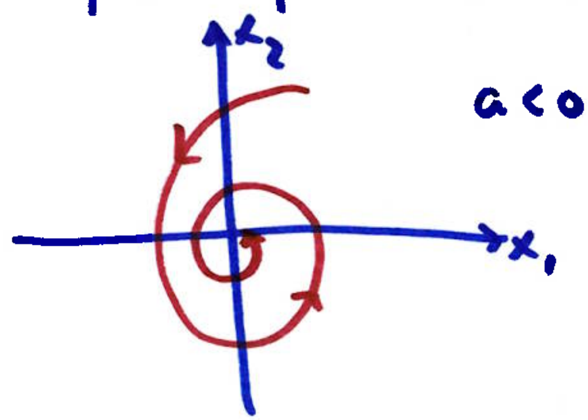
true eigenvec

if $\lambda = a + bi$

$\vec{x} = c_1 e^{at}\vec{u} + c_2 e^{at}\vec{v}$

$\vec{u}, \vec{v}$ are real and imag. parts of one solution





$$\vec{x}' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \vec{x}$$

$$\lambda = -2+i \;,\; -2-i$$

$$\vec{v} = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}, \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$$

$$e^{ibt} = \cos(bt) + i\sin(bt)$$

$$e^{it} = \cos(t) + i\sin(t)$$

pick a pair, form one solution

$$e^{(-2+i)t} \begin{bmatrix} 2 \\ -1+i \end{bmatrix} = e^{-2t} e^{it} \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

$$e^{-2t} (\cos t + i \sin t) \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 2\cos t + i\, 2\sin(t) \\ -\cos t - \sin t - i \sin t + i \cos t \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 2\cos t \\ -\cos t - \sin t \end{bmatrix} + i\, e^{-2t} \begin{bmatrix} 2\sin t \\ \cos t - \sin t \end{bmatrix}$$

form solutions w/ these

nonhomogeneous: undetermined coeff

variation of parameters

undetermined coeff: follow the form of right side

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{x}_c = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

right side: $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ so, $\vec{y}_p = \vec{a}$

fundamental solutions as columns

variation of parameters: $\vec{x} = \Psi \vec{u}$

$$= \begin{bmatrix} e^{3t} & e^{-2t} \\ e^{3t} & -4e^{-2t} \end{bmatrix} \vec{u}$$

where $\Psi \vec{u}' = \vec{g}$ right side