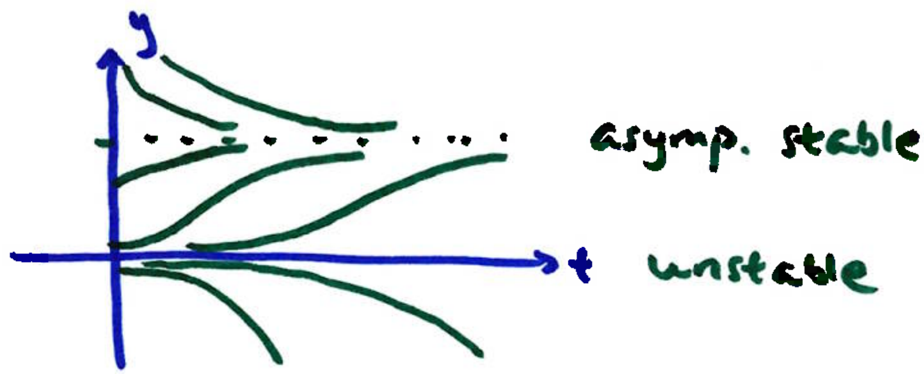
2.5 (continued)

asymptotically stable: neighboring solutions converge onto this equilibrium

unstable: solutions run away from this



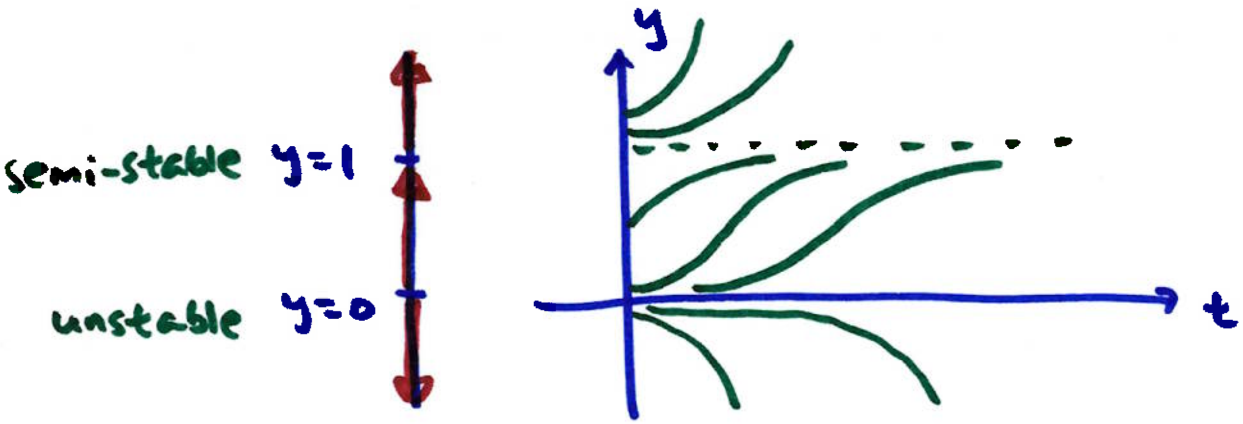
another type

$$\frac{dy}{dt} = y(y-1)^2 = f(y)$$

equilibrium / critical point: $f(y)=0$

here, $y=0$, $y=1$

phase line : $\quad - \quad 0 \quad + \quad 0 \quad +$ $\quad$ sign of $f(y) = y(y-1)^2$

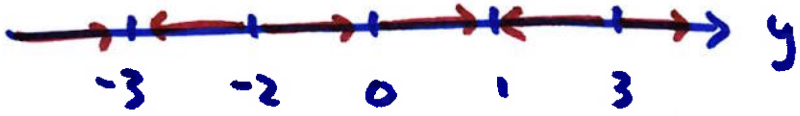


semi-stable : stable from one side only

<u>example</u> $\quad y' = y^2 (y-1)(y+2)(y^2-9)$

critical pts : 0, 1, -2, -3, 3

+ 0 − 0 + 0 + 0 − 0 + sign of $f(y) = y^2(y-1)(y+2)(y^2-9)$



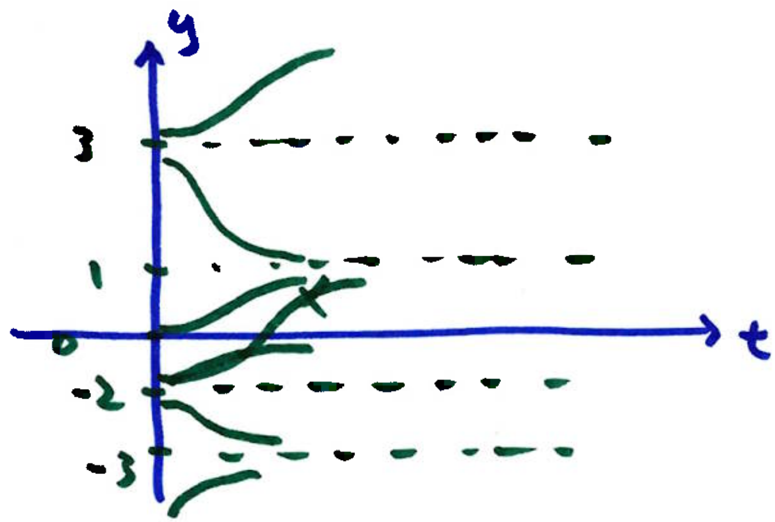
-3 : asymp stable

-2 : unstable

0 : semi-stable

1 : stable

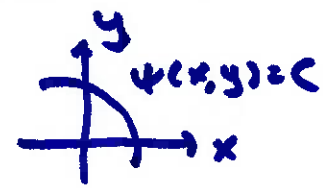
3 : unstable



2.6 Exact Diff. Eqs.

implicit function of x

from calculus, we saw level curves $\psi(x, y) = C$



differentiate $\psi(x,y)$ with respect to x

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0$$

let $\frac{\partial \psi}{\partial x}(x,y) = M(x,y)$ and $\frac{\partial \psi}{\partial y}(x,y) = N(x,y)$

then we have $\boxed{M(x,y) + N(x,y)\,y' = 0}$

if ψ has at least up to 2nd-order derivatives being continuous, then $\psi_{xy} = \psi_{yx}$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{or} \quad M_y = N_x}$$

a diff. eq. of the form

$M(x,y) + N(x,y)\,y' = 0$ such that $M_y = N_x$

is called an <u>exact diff. eq.</u>

sometimes written in differential form as $M\,dx + N\,dy = 0$

so, there <u>must</u> exist a function $\psi(x,y) = C$ which we can

use as an <u>implicit solution</u>.

<u>example</u>

$$\underbrace{(3x^2 + 2y^2)}_{M} + \underbrace{(4xy + 6y^2)}_{N}\,y' = 0$$

attached to dx or nothing in the "usual" form

attached to y' or dy in differential form it looks exact, but is it?

check: $M_y = N_x$ if so, it is exact

if not, it might be or not

$M = 3x^2 + 2y^2$ $M_y = 4y$

$N = 4xy + 6y^2$ $N_x = 4y$

$\left.\begin{matrix} M_y = 4y \\ N_x = 4y \end{matrix}\right\}$ equal, so is exact

so there is a $\psi(x,y)$ such that $\psi_x = M$

$\psi_y = N$

$$\left.\begin{aligned} \psi_x &= 3x^2 + 2y^2 \\ \psi_y &= 4xy + 6y^2 \end{aligned}\right\} \text{recover } \psi \text{ by integration}$$

let's integrate ψ_x with respect to x

$$\psi(x,y) = \int \psi_x \, d\underline{\underline{x}} = \int M \, d\underline{\underline{x}} = \int (3x^2 + 2y^2) \, d\underline{\underline{x}}$$

variable is x
so y is "constant"

$$\psi(x,y) = x^3 + 2y^2x + h(y)$$

disappears when $\frac{\partial}{\partial x}$ and
no trace of it shows up in M
so could be a constant but
more generally a function of y

whatever $h(y)$ is, the partial of $\psi(x,y) = x^3 + 2y^2x + h(y)$
MUST be equal to N because $N = \psi_y$

$$\frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\left(x^3 + 2y^2x + h(y)\right) = \underbrace{4xy + 6y^2}_{\text{given } N}$$

$$4yx + h'(y) = $$

so, $h'(y) = 6y^2$ $\quad h(y) = 2y^3 + c_1$

choose $c_1 = 0$
usually

so, $\psi(x,y) = x^3 + 2xy^2 + 2y^3 + C_1$

solution is implicitly defined $\psi(x,y) = C$

so we get $\boxed{x^3 + 2xy^2 + 2y^3 = C}$

many separable eqs. are exact

$$\frac{dy}{dx} = \frac{x}{y} \rightarrow y\frac{dy}{dx} = x \rightarrow x - yy' = 0$$

$M = x$

$N = -y$

$M_y = 0 \qquad N_x = 0$

but not all sep. are exact

$$\frac{dy}{dx} = \frac{y}{x} \rightarrow y - xy' = 0$$

$M = y \qquad N = -x$

$M_y = 1 \qquad N_x = -1$

NOT exact