2.6 (continued)

exact: $M(x,y) + N(x,y)y' = 0$ such that $M_y = N_x$

solution is $\psi(x,y) = C$ where $\psi_x = M$ and $\psi_y = N$

sometimes, we can make an equation exact by multiplying by an integrating factor like with linear eqs. (same idea but different process)

$$\underbrace{e^x}_{M} + \underbrace{(e^x \cot y + 2y \csc y)}_{N}y' = 0$$

$M_y = 0 \qquad N_x = e^x \cot y \qquad M_y \neq N_x$ so not exact

but notice if we multiply both sides by $\sin y$

$$e^x \sin y + (e^x \cos y + 2y)y' = 0$$

new $M = e^x \sin y$ new $N = e^x \cos y + 2y$

$M_y = e^x \cos y$ $N_x = e^x \cos y$ exact !

now find ψ such that $\psi_x = \cancel{e^x \cos y}$ $e^x \sin y$, $\psi_y = e^x \cos y + 2y$

How to find μ ~~for exac~~ to make eq. exact ?

$$M(x,y) + N(x,y)y' = 0 \quad \text{but} \quad M_y \neq N_x$$

multiply by $\mu(x,y)$

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$$

to be exact

$$\frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right]$$

$$\mu(x,y)M_y(x,y) + \mu_y(x,y)M(x,y)$$
$$= \mu(x,y)N_x(x,y) + \mu_x(x,y)N(x,y)$$

$$\mu M_y + \mu_y M = \mu N_x + \mu_x N$$

$$\boxed{\mu_y M - \mu_x N + (M_y - N_x)\mu = 0}$$

Solve this partial diff. eq. (PDE) for μ

not easy : much harder than the original (non exact) ODE.

ANY μ that satisfies this eq. can be used to make the eq. exact

assume μ is $\mu(x)$ or $\mu(y)$ only

than one of μ_x or μ_y goes away.

for example, if we say $\mu = \mu(y)$

$$\frac{d\mu}{dy} M + (M_y - N_x)\mu = 0 \qquad \text{ODE !}$$

$$\frac{d\mu}{dy} = -\frac{(M_y - N_x)\mu}{M} \qquad \text{separable!}$$

<u>example</u> $(e^{2x}+y-1)-y'=0$

$M=e^{2x}+y-1 \qquad N=-1$

$M_y=1 \quad N_x=0$ not exact

find a μ to make it exact

$\mu(e^{2x}+y-1)-\mu y'=0$ is exact

$$\frac{\partial}{\partial y}\left[\mu(e^{2x}+y-1)\right]=\frac{\partial}{\partial x}\left(-\mu\right)$$

make a choice: $\mu=\mu(x)$ or $\mu=\mu(y)$

let's go with $\mu=\mu(x)$

on the left, μ is "constant"

so, $\mu=-\dfrac{d\mu}{dx}$ separable

$$\frac{1}{\mu}\, d\mu = -dx$$

$\vdots$

$\mu = Ce^{-x}$ since μ is multiplied to both sides of original eq. C is arbitrary

$\mu = e^{-x}$ choose any $C \neq 0$

back to $(e^{2x}+y-1) - y' = 0$

multiply by $\mu = e^{-x}$

$$e^{-x}(e^{2x}+y-1) - e^{-x}y' = 0$$

$$(e^{x}+e^{-x}y-e^{-x}) + (-e^{-x})y' = 0$$

new $M = e^{x}+e^{-x}y-e^{-x}$ $\quad M_y = e^{-x}$

$N = -e^{-x}$ $\quad N_x = e^{-x}$

find Ψ : $\Psi_x = e^{x}+e^{-x}y-e^{-x}$

$\Psi_y = -e^{-x}$

this time, let's integrate $\psi_y = -e^{-x}$

$$\psi = \int -e^{-x}\, dy = -e^{-x} y + h(x)$$

$\psi_x = e^{-x} y + h'(x)$ must match $M = e^{x} + y e^{-x} - e^{-x}$

$$h'(x) = e^{x} - e^{-x}$$

$h(x) = e^{x} + e^{-x}$ no need for C here

so, $\psi = -e^{-x} y + e^{x} + e^{-x}$

solution : $\psi = c$ $\boxed{e^{x} - e^{-x} - e^{-x} y = c}$

$$e^{2x} - 1 - y = Ce^{x}$$

$$y = e^{2x} - 1 + Ce^{x}$$

let's look at the original eq. again

$$(e^{2x}+y-1)-y'=0$$

$$y'-y=e^{2x}-1$$ hold up! it's linear!

$$\mu=e^{\int -x\,dx}=e^{-x}$$

$$e^{-x}(y'-y)=e^{-x}(e^{2x}-1)$$

$$\frac{d}{dx}(e^{-x}y)=e^{x}-e^{-x}$$

$$e^{-x}y=e^{x}+e^{-x}+c$$

$$y=e^{2x}+1+ce^{x}$$ same!

ALWAYS identify the type of eq. first