2.7 Numerical Approx: Euler's Method

$y' = f(t, y)$ if not linear, separable, exact or homogeneous,

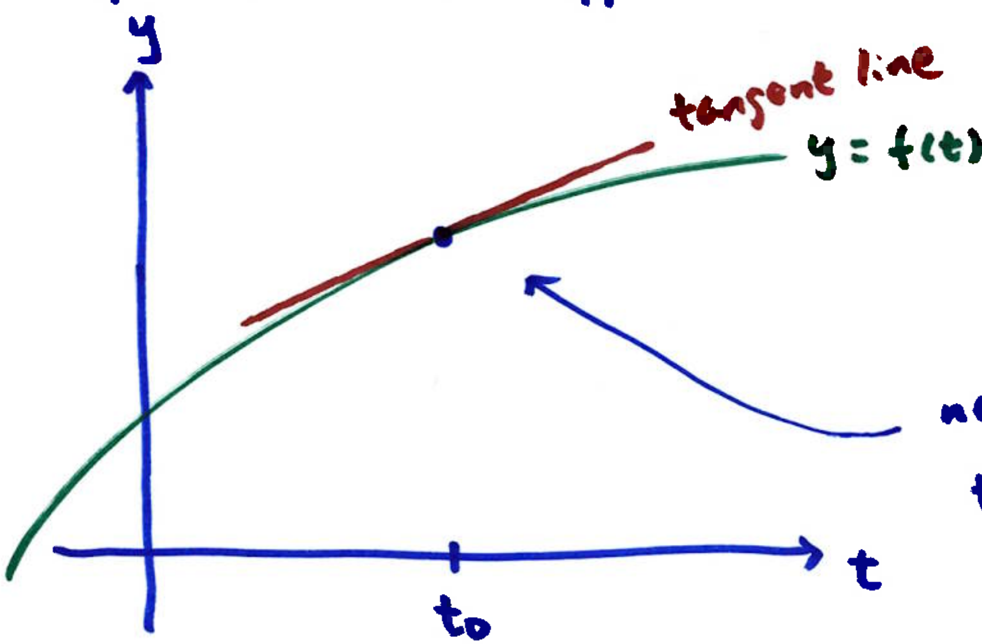
what do we do?

numerical methods: $(t_0, y_0), (t_1, y_1), (t_2, y_2), \ldots$ instead of

a function $y = f(t, y)$

Euler's method is also called the tangent line method

from calculus, we saw linear approx.

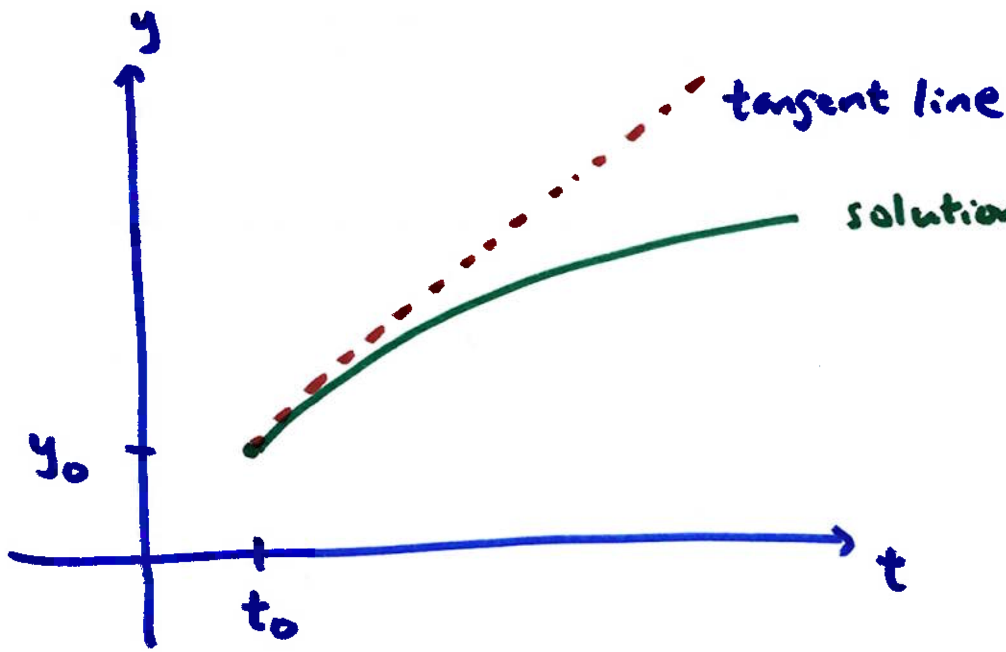


near $t = t_0$,

tangent line $\approx$ true curve

we will use the same idea, except now we don't have the solution y but we have y'

$y(t_0) = y_0$ is always known



solution : y we don't have this

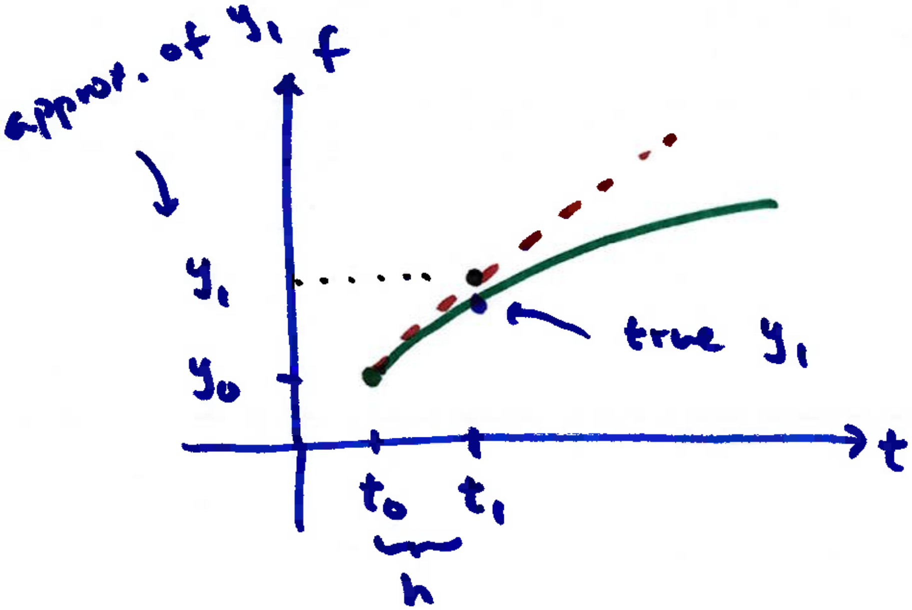
we know y' at any (t, y)

tangent line at (t_0, y_0) is $y - y_0 = f(t_0, y_0)(t - t_0)$

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

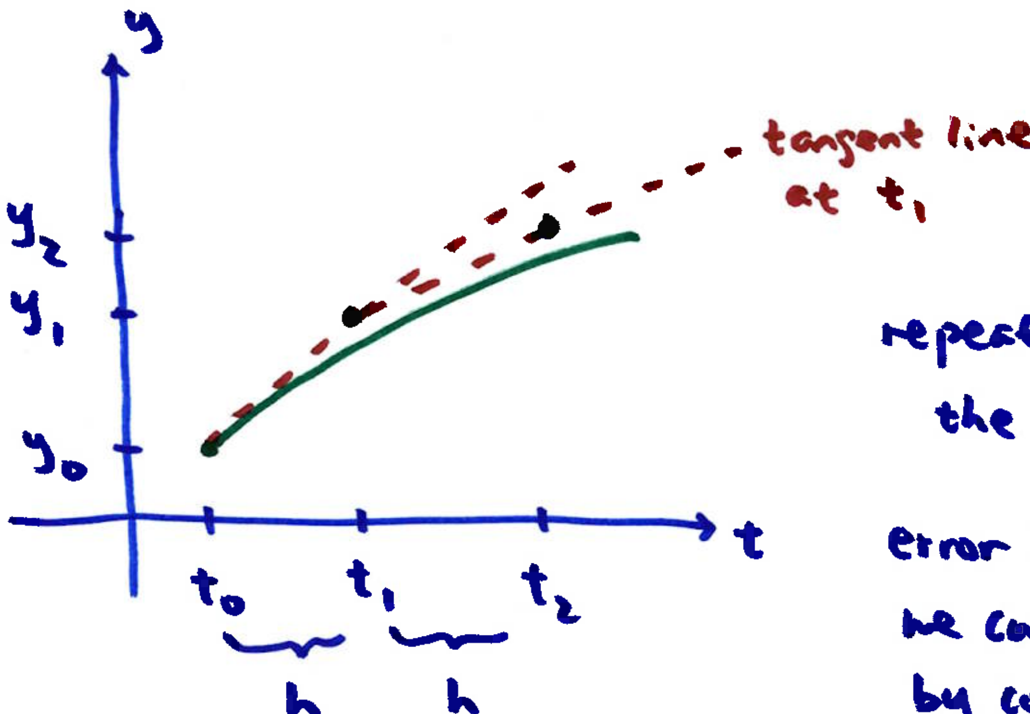
travel on tangent line to a new value of t

decide the change in t : $\Delta t = h$



if h is "small" then

approx. $y_1 \approx$ true y_1



repeat until we reach the target t

error in every step, but we control the accuracy by controling h

<u>example</u> $y' = 2y - 3t \quad y(0) = 1$

estimate $y(0.5)$ using a step size of $h = 0.25$

$t_0 = 0$

$y_0 = 1$

$t_1 = t_0 + h = 0 + 0.25 = 0.25$

$$y_1 = y_0 + f(\underbrace{t_0, y_0}_{\text{previous } (t,y) \text{ to calculate slope}})\underbrace{(t_1 - t_0)}_{h}$$

$$= 1 + [2(1) - 3(0)](0.25) = 1.5 \quad \rightarrow y(0.25) \approx 1.5$$

$t_2 = t_1 + h = 0.25 + 0.25 = 0.5$ target t so this is the final step

$$y_2 = y_1 + f(t_1, y_1)h$$

$$= 1.5 + [2(1.5) - 3(0.25)](0.25) = 2.0625$$

$y(0.5) \approx 2.0625$

how good is that estimate?

$y' = 2y - 3t$ is linear, we can solve exactly

$\vdots$

$y = \frac{3}{4}(2t+1) + \frac{1}{4}e^{2t}$

true $y(0.5) = \frac{3}{4}(2) + \frac{1}{4}e \approx 2.1796$

estimate w/ $h = 0.25$ gave us $y(0.5) \approx 2.0625$ (5% error)

want better estimate? use smaller h (more steps)

if we used $h = 0.01$ (50 steps!)

$y(0.5) \approx 2.1729$ (0.3% error)

3.1 Homogeneous Diff. Eqs. with Constant Coefficients

focus on 2nd order : $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$

(extends to n^{th} order easily)

linear: $f(t, y, y')$ is linear in y, y' and anything to do w/ y

form: $P(t)y'' + Q(t)y' + R(t)y = G(t)$

no y of any kind

Standard form: $y'' + p(t)y' + q(t)y = g(t)$

"homogeneous" in this context means $g(t) = 0$

(more common meaning of this word)

constant coefficient: $ay'' + by' + cy = 0$ a, b, c constants

$ay'' + by' + cy = 0$

y is related to its own 1st and 2nd derivatives by constant multiples

→ exponential, sine & cosine

↑ focus for now

so, we want solution(s) that look like e^{rt} r : constant

example $y'' + 2y' - 3y = 0$

solution :

$$\left.\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}\right\}$$ must satisfy the diff. eq.

$$r^2 e^{rt} + 2re^{rt} - 3e^{rt} = 0$$

$$e^{rt}(r^2 + 2r - 3) = 0$$

since $e^{rt} \neq 0$, this means

$$\boxed{r^2 + 2r - 3 = 0}$$ characteristic eq.

for $y = e^{rt}$ to be a solution, r must satisfy $r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$ $\quad r = -3,\ r = 1$ $\quad$ two r $\rightarrow$ two solutions

solutions: $\left.\begin{array}{l} y_1 = e^{-3t} \\ y_2 = e^{t} \end{array}\right\}$ fundamental solutions

Since the diff. eq. is linear, any linear combination of the fundamental solutions is also a solution.

$\boxed{y = c_1 e^{-3t} + c_2 e^{t}}$ general solution

c_1, c_2 depend on initial conditions

↳ two needed to determine c_1 and c_2 uniquely

typically $y(t_0) = y_0,\ y'(t_0) = y_0'$