3.1 (continued)

$ay'' + by' + cy = 0 \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$

solutions: $y = e^{rt}$ $\quad$ r: constant

Sub into the diff. eq. $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$

$e^{rt}(ar^2 + br + c) = 0 \quad e^{rt} \neq 0$

$\boxed{ar^2 + br + c = 0}$ characteristic eq.

notice the structure compared to

$ay'' + by' + cy = 0$

two solutions: $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ $\quad$ r_1, r_2 are roots of char. eq.

general solution: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

c_1, c_2 come from initial conditions

<u>example</u> $y'' + 8y' - 9y = 0 \quad y(0) = 1, \ y'(0) = 0$

char. eq: $r^2 + 8r - 9 = 0$

$(r+9)(r-1) = 0$

$r = -9, \ r = 1$

general solution: $y = c_1 e^{-9t} + c_2 e^{t}$

find c_1, c_2 from $y(0) = 1, \ y'(0) = 0$

$y(0) = 1 \longrightarrow 1 = c_1 + c_2$

to use $y'(0) = 0$, must have y'

from $y = c_1 e^{-9t} + c_2 e^t$ we get

$$y' = \underbrace{-9c_1}_{\text{cannot call this } c_1 \text{ since we care about value of } c_1} e^{-9t} + c_2 e^t$$

$$y'(0)=0 \rightarrow \left.\begin{aligned} 0 &= -9c_1 + c_2 \\ 1 &= c_1 + c_2 \end{aligned}\right\}$$ solve simultaneously

mult. by 9: $9 = 9c_1 + 9c_2$

add to $0 = -9c_1 + c_2$

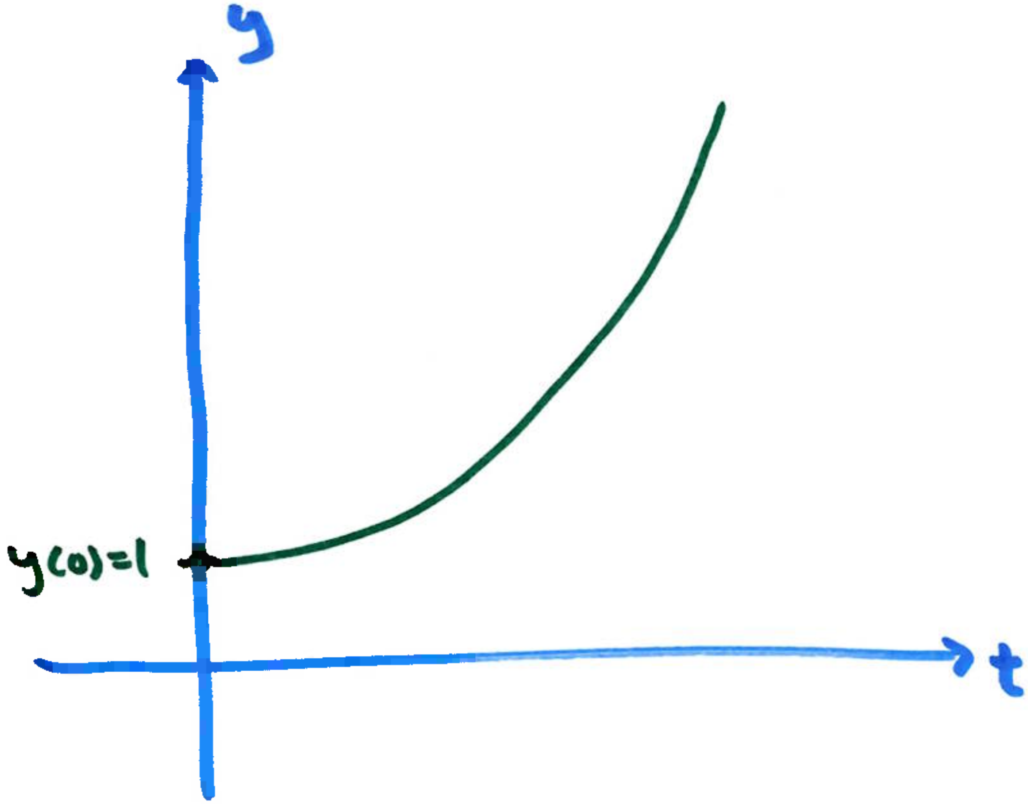
$$9 = 10c_2 \qquad c_2 = \frac{9}{10}$$

then $c_1 = 1 - c_2 = \frac{1}{10}$

particular solution: $\boxed{y = \frac{1}{10}e^{-9t} + \frac{9}{10}e^{t}}$

$$y' = -\frac{9}{10}e^{-9t} + \frac{9}{10}e^{t}$$

sketch



$y(t_0) = y_0$: initial position

$y'(t_0) = y_0'$: initial slope

$y(0) = 1$

$y'(0) = 0$ → slope of curve at $t=0$ (flat at $t=0$)

$$y' = -\underbrace{\frac{9}{10}e^{-9t}}_{<\, e^t \text{ as } t \text{ increases}} + \frac{9}{10}e^{t}$$

$0 \leq \text{slope} < \frac{9}{10}$ as $t \to \infty$

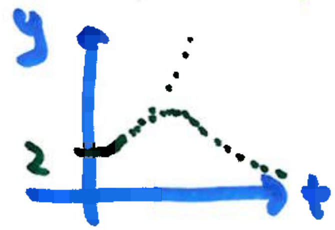
$$\lim_{t\to\infty} e^{-9t} = 0$$

$$\lim_{t\to\infty} y' = \lim_{t\to\infty} \frac{9}{10}e^{t}$$

$$\lim_{t\to\infty} y = \frac{9}{10}e^{t}$$

<u>example</u> $y'' + 5y' + 6y = 0 \quad y(0) = 2 \quad \underbrace{y'(0) = \beta}_{} \quad (\beta > 0)$

initial slope is pos.



Char. eq: $r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$r = -3, \; r = -2$

$y = C_1 e^{-3t} + C_2 e^{-2t}$

$y' = -3C_1 e^{-3t} - 2C_2 e^{-2t}$

$y(0) = 2 \rightarrow 2 = C_1 + C_2$

$y'(0) = \beta \rightarrow \beta = -3C_1 - 2C_2$

Since $\lim_{t\to\infty} e^{-3t} = \lim_{t\to\infty} e^{-2t} = 0$

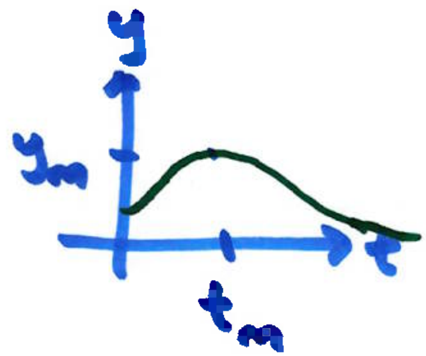
$\lim_{t\to\infty} y = 0$

$y'(0) > 0$, so initial go up, reach a max, then approach 0

after solving the system,

$$C_1 = -(4+\beta) \quad , \quad C_2 = \beta + 6$$

$$y = -(4+\beta)e^{-3t} + (\beta+6)e^{-2t}$$



time to reach max y ?

$$y' = 3(4+\beta)e^{-3t} - 2(\beta+6)e^{-2t}$$

max $y \rightarrow y' = 0$

$$3(4+\beta)e^{-3t} - 2(\beta+6)e^{-2t} = 0$$

$$3(4+\beta)e^{-3t} = 2(\beta+6)e^{-2t}$$

$$3(4+\beta) = 2(\beta+6)e^{t}$$

$$e^{t} = \frac{3(4+\beta)}{2(\beta+6)}$$

$$\boxed{t_m = \ln\left(\frac{12+3\beta}{2\beta+12}\right)}$$

note

$$\lim_{\beta\to\infty} t_m = \ln\left(\frac{3}{2}\right)$$

Sub t_m into y to find y_m

after some algebra,

$$\boxed{y_m = \frac{4}{27}\frac{(6+\beta)^3}{(4+\beta)^2}}$$

3.2 Fundamental Solutions of Linear Homogeneous Eqs.

Solutions of $ay'' + by' + cy = 0 \quad y(t_0) = y_0, \; y'(t_0) = y_0'$

are always unique

but what about $y'' + p(t)y' + q(t)y = g(t)$

very similar to $y' + p(t)y = g(t), \quad y(t_0) = y_0$

when p, g are both continuous

and containing $y(t_0) = y_0$

2nd order: same idea

$$y'' + p(t)y' + q(t)y = g(t)$$

find interval of t where all

are continuous and containing t_0