3.2 (continued)

$y'' + p(t)y' + q(t)y = g(t) \qquad y(t_0) = y_0, \ y'(t_0) = y_0'$

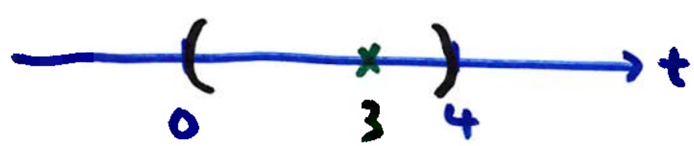
unique solution on interval containing t_0 where

$p(t)$, $q(t)$, $g(t)$ are continuous

for example, $t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0, \ y'(3) = -1$

$$y'' + \underbrace{\frac{3t}{t(t-4)}}_{p} y' + \underbrace{\frac{4}{t(t-4)}}_{q} y = \underbrace{\frac{2}{t(t-4)}}_{g}$$

p, q, g continuous $t \neq 0, \ t \neq 4$



there is one and only one solution on $0 < t < 4$

2nd-order : two fundamental solutions y_1, y_2

unique solution means there is one and only one linear combination of y_1 and y_2 for a given set of initial conditions

general solution: $y = c_1 y_1 + c_2 y_2$

why is this a solution in the first place?

$$y'' + py' + qy = 0$$

if y_1 and y_2 are solutions, then

$$y_1'' + py_1' + qy_1 = 0$$

$$y_2'' + py_2' + qy_2 = 0$$

let's sub $y = c_1 y_1 + c_2 y_2$ in to see it is indeed a solution

$$(c_1 y_1'' + c_2 y_2'') + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2) = 0$$

$$c_1 (\underbrace{y_1'' + py_1' + qy_1}_{0}) + c_2 (\underbrace{y_2'' + py_2' + qy_2}_{0}) = 0$$

so, $y = c_1 y_1 + c_2 y_2$ $\underline{\underline{\text{is}}}$ a solution

---

$$y'' + py' + qy = 0 \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Solutions y_1, y_2

general solution $y = c_1 y_1 + c_2 y_2$

under what conditions can we find c_1 and c_2 for any given ~~y_0~~ y_0, y_0' ?

$$y = c_1 y_1 + c_2 y_2 \qquad y' = c_1 y_1' + c_2 y_2'$$

$$\left.\begin{aligned} y(t_0) = y_0 &\longrightarrow c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ y'(t_0) = y_0' &\longrightarrow c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{aligned}\right\}$$

can we find a solution (unique) for c_1, c_2 w/ any y_0, y_0' ?

rewrite in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

unique solution if $\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$ is invertible

if so,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \begin{bmatrix} y_2'(t_0) & -y_2(t_0) \\ -y_1'(t_0) & y_1(t_0) \end{bmatrix} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

$$\boxed{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0}$$

we ~~called~~ call that determinant the Wronskian

$$W[y_1, y_2](t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

as long as $W[y_1, y_2](t_0) \neq 0$, we can always uniquely find c_1, c_2 for any y_0, y_0'

$y'' + 8y' - 9y = 0$

$r^2 + 8r - 9 = 0 \qquad (r+9)(r-1) = 0 \qquad r = -9,\ r = 1$

$y_1 = e^{-9t} \qquad y_2 = e^{t}$

$$W[y_1, y_2](t_0) = \begin{vmatrix} e^{-9t} & e^{t} \\ -9e^{-9t} & e^{t} \end{vmatrix} = e^{-8t} + 9e^{-8t} = 10e^{-8t}$$

notice $e^{-8t} \neq 0$ for any t

$\longrightarrow$ we can find a unique set of c_1, c_2

for ANY $y(t_0) = y_0,\ y'(t_0) = y_0'$

do we need to know y_1 and y_2 to find the Wronskian?

surprisingly, no!

$y'' + py' + qy = 0$ has y_1, y_2 as fundamental solutions

then $y_1'' + py_1' + qy_1 = 0$

$$y_2'' + py_2' + qy_2 = 0$$

multiply 1st by $-y_2$ and 2nd by y_1

$$-y_2 y_1'' - p y_2 y_1' - q y_2 y_1 = 0$$

$$y_1 y_2'' + p y_1 y_2' + q y_2 y_1 = 0$$

now add them

W

$$\underbrace{(y_1 y_2'' - y_2 y_1'')}_{W'} + p \underbrace{(y_1 y_2' - y_1' y_2)}_{?} = 0$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W' = y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2$$

$$W' = y_1 y_2'' - y_2 y_1''$$

we end up with

$$\boxed{w' + pw = 0}$$

linear and separable

$$\frac{dw}{dt} = -pw$$

$$\frac{1}{w}\,dw = -p\,dt$$

$$\ln|w| = \cancel{-pt + c} \quad \int -p\,dt + c$$

$$|w| = \cancel{e^{c} \cdot e^{-pt}} = e^{c} \cdot e^{\int -p\,dt}$$

$$\boxed{w = c\,e^{\int -p\,dt}}$$

Abel's Formula or Abel's Identity

$$\cos(t)\, y'' + \sin(t)\, y' - t y = 0$$

$$y'' + \underbrace{\frac{\sin(t)}{\cos(t)}}_{p} y' - \frac{t}{\cos(t)}\, y = 0$$

$$W = C e^{-\int \frac{\sin(t)}{\cos(t)} dt} = C e^{+\ln \cos(t)} = \cancel{Ce}^{1}$$

$$= C \cdot \cos(t)$$