3.3 Complex Roots of the Char. Eq.

$\underbrace{y''+y=0}$ $\quad r^2+1=0 \quad r=i,-i \quad i^2=-1 \quad i=\sqrt{-1}$

y such that it is the negative of its 2nd derivative

$\to$ $y=\cos(t)$, $y=\sin(t)$

Solution: $y=e^{rt}$ $\qquad \underbrace{y_1=e^{it} \qquad y_2=e^{-it}}$

$\cos(t)$, $\sin(t)$ ~~sth~~ somehow?

how does this work?

$$e^t = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots+\frac{t^n}{n!}+\dots$$

$$e^{(it)} = 1+it-\frac{t^2}{2!}-\frac{it^3}{3!}+\frac{t^4}{4!}+\frac{it^5}{5!}+\frac{-t^6}{6!}+\dots$$

$$= \left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\dots\right)+i\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+\dots\right)$$

$$\boxed{e^{it} = \cos(t)+i\sin(t)}$$

Euler's Formula

$y'' + y = 0 \qquad r = \pm i$

$y_1 = e^{it} = \cos(t) + i\sin(t)$

$y_2 = e^{-it} = e^{i(-t)} = \cos(-t) + i\sin(-t)$

$\qquad = \cos(t) - i\sin(t)$

general solution: $\underbrace{y = c_1 y_1 + c_2 y_2}_{\text{expression contains } i}$ is real for real-valued $y(t_0) = y_0,\ y'(t_0) = y_0'$

usually, we don't want i in our real-valued solutions

notice the real part and imaginary part ($\cos(t)$, $\sin(t)$) are themselves solutions to $y'' + y = 0$

so, it's equally valid to say $y_1 = \cos(t)$, $y_2 = \sin(t)$

→ use the real and imaginary parts of either solution as the fundamental solutions __example__ $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

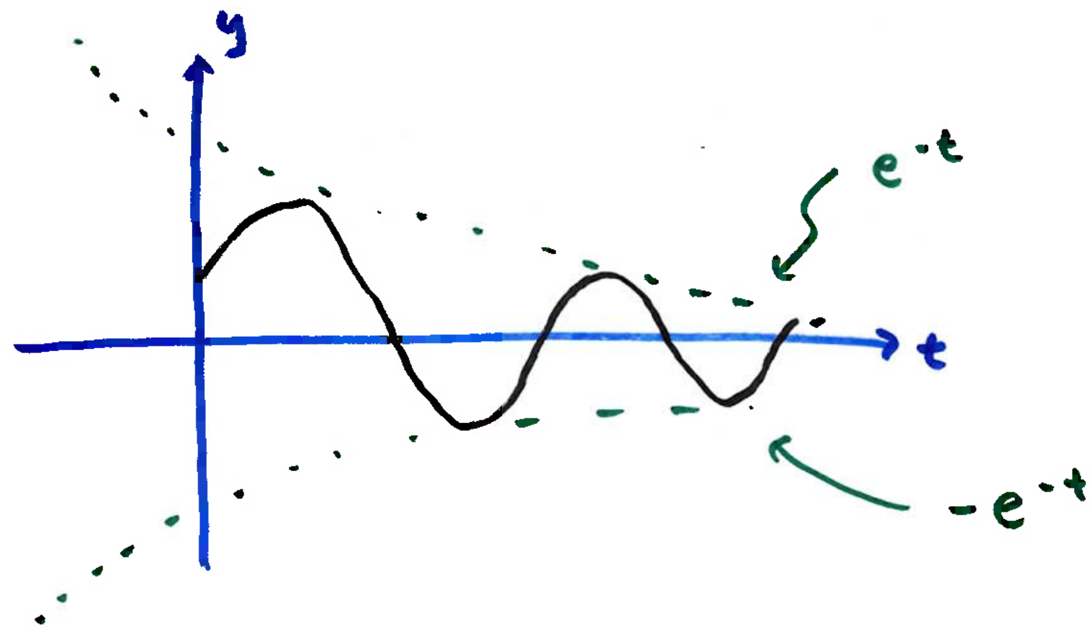
form one solution using e^{rt}, extract real and imag. parts as fundamental solutions

$$e^{(-1+i)t} = e^{-t} e^{it} = e^{-t}(\cos(t) + i \sin(t))$$
$$= \boxed{e^{-t}\cos(t)} + i \boxed{e^{-t}\sin(t)}$$

so, fundamental solutions are $y_1 = e^{-t}\cos(t)$

$y_2 = e^{-t}\sin(t)$

general solution $\boxed{y = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)}$



y
t
e-t
-e-t

3.4 Repeated Roots, Reduction of Order

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0 \qquad (r-1)(r-1) = 0 \qquad r = 1, 1$

$y_1 = e^{r_1 t} = e^t$

$y_2 = e^{r_2 t} = e^t$?

$W[y_1, y_2] = \begin{vmatrix} e^t & e^t \\ e^t & e^t \end{vmatrix} = 0$

cannot find c_1, c_2 to meet the initial condition to find a unique solution

y_2 must NOT be the same as y_1

$y_2 = ?$

$$\boxed{\begin{array}{l}\text{Reduction of Order (due to D'Alembert)} \\ \\ y_2 = v(t)\, y_1\end{array}}$$

to find $v(t)$, sub $y_2 = v(t)\, y_1$ into the diff. eq.

$$y'' - 2y' + y = 0$$

$$y_1 = e^t$$

$$y_2 = ve^t$$

$$y_2' = ve^t + v'e^t$$

$$y_2'' = ve^t + v'e^t + v'e^t + v''e^t$$

$$= ve^t + 2v'e^t + v''e^t$$

$$\cancel{ve^t} + \cancel{2v'e^t} + v''e^t - \cancel{2ve^t} - \cancel{2v'e^t} + \cancel{ve^t} = 0$$

$$v'' = 0$$

$v = at + b$ we can use ANY a, b $(a \neq 0)$ as v for $y_2 = vy_1$

let's use $v = t$

then $y_1 = e^t$
$y_2 = te^t$

$$W[y_1, y_2] = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix}$$

$$= te^{2t} + e^{2t} - te^{2t} = e^{2t} \neq 0 \text{ (good!)}$$

if the differential eq. has constant coefficient

$y_2 = t\,y_1$ is ALWAYS a valid 2nd solution.

but Reduction of Order can be used to find missing solution even for non-constant coefficients

$t^2 y'' + t y' - 9y = 0$ (this is an example of Euler's equation)

has one solution $y_1 = t^3$ (somehow we found this)

$$y_2 = ?$$

$$y_2 = v y_1 = v t^3$$

$$y_2' = 3vt^2 + v't^3$$

$$y_2'' = 6vt + 6v't^2 + v''t^3$$

$$t^2(6vt + 6v't^2 + v''t^3) + t(3vt^2 + v't^3) - 9(vt^3) = 0$$

$\vdots$

$$t^5 v'' + 7v't^4 = 0$$

$v'' = -7v't^{-1}$ $\qquad$ $v'' = \dfrac{d(v')}{dt}$

$\dfrac{d(v')}{dt} = -\dfrac{7v'}{t}$ $\qquad$ separable in v' and t

solve for v' ... $\quad v' = c\,t^{-7}$

then $v = -\frac{c}{6}t^{-6} + d$ $\qquad$ choose $c = -6,\ d = 0$

$v = t^{-6}$

so $y_2 = v y_1 = t^{-6} \cdot t^{3} = t^{-3}$