# 3.5 Nonhomogeneous Eqs.: Undetermined Coefficients

$ay'' + by' + cy = f(t)$

if $f(t) = 0$ (homogeneous) then $y = c_1 y_1 + c_2 y_2$

eq. is linear, so principle of superpositition applies

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{if the eq. were homogeneous } (f(t)=0)} + Y(t)$$

due to the nonhomogeneous part ($f(t)$)
"particular solution"

if the eq. were homogeneous ($f(t)=0$)
"complementary solution"

to find $Y(t)$, one method is undetermined coefficients

basic idea: $Y(t)$ resembles $f(t)$

if $f(t)$ is polynomial, so is $Y(t)$
if $f(t)$ is exponential, so is $Y(t)$
if $f(t)$ is cosine or sine, so is $Y(t)$

**example** $y'' - y' - 2y = -2t + 4t^2$

find complementary solution: solve $y'' - y' - 2y = 0$

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = -1, \quad r = 2$$

$$y_c = c_1 e^{-t} + c_2 e^{2t}$$

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + Y(t)$$

$Y(t)$ will look like $f(t)$

$f(t) = 4t^2 - 2t$ → 2nd-order polynomial

copy the <u>form</u> for $Y(t)$

$$Y(t) = At^2 + Bt + C$$

undetermined coefficients

$Y(t)$ <u>is</u> a solution of $y'' - y' - 2y = -2t + 4t^2$

so it must satisfy the diff. eq.

plug $Y$ into diff. eq. in place of $y$

$$\left.\begin{aligned} Y &= At^2 + Bt + C \\ Y' &= 2At + B \\ Y'' &= 2A \end{aligned}\right\} \quad y'' - y' - 2y = -2t + 4t^2$$

$$2A - 2At - B - 2At^2 - 2Bt - 2C = -2t + 4t^2$$

$$\underline{\underline{-2A}}t^2 + \underline{(-2A - 2B)}t + \underbrace{(2A - B - 2C)} = \underline{\underline{4}}t^2 \underline{-2}t + \underbrace{0}$$

$$\begin{aligned} -2A &= 4 & &\longrightarrow & A &= -2 \\ -2A - 2B &= -2 & &\longrightarrow & B &= 3 \\ 2A - B - 2C &= 0 & &\longrightarrow & C &= -7/2 \end{aligned}$$

so, the solution is

$$\boxed{y(t) = \underbrace{c_1 e^{-t} + c_2 e^{2t}}_{\substack{\text{complementary} \\ \text{affected by initial} \\ \text{conditions}}} \underbrace{- 2t^2 + 3t - \frac{7}{2}}_{\substack{\text{particular} \\ \text{affected by } f(t) \text{ only}}}}$$

**example** $y'' - y' - 2y = 10e^{t}$

same left side, so $y_c = c_1 e^{-t} + c_2 e^{2t}$

$Y(t)$ matches form of $f(t) = 10e^{t}$

$$\left.\begin{aligned} Y(t) &= Ae^{t} \\ Y'(t) &= Ae^{t} \\ Y''(t) &= Ae^{t} \end{aligned}\right\} \text{ into diff. eq.}$$

$$Ae^{t} - Ae^{t} - 2Ae^{t} = 10e^{t}$$

$$-2A = 10 \qquad A = -5$$

solution: $\boxed{y = c_1 e^{-t} + c_2 e^{2t} - 5e^{t}}$

what about $y'' - y' - 2y = -2t + 4t^2 + 10e^{t}$ ?

$$y = c_1 e^{-t} + c_2 e^{2t} + \underbrace{-2t^2 + 3t - \tfrac{7}{2}}_{\text{from } -2t+4t^2} \underbrace{-5e^{t}}_{\text{from } 10e^{t}}$$

notice if $f(t)$ is polynomial, then $Y, Y', Y''$ remain polynomial

if $f(t)$ is exponential, same story

but if $Y = A\cos(t)$

$Y' = -A\sin(t)$ no longer cosine!

its <u>form</u> changed

for undetermined coefficients to work, the <u>form</u> must remain the same

if $f(t) = \cos(t)$

we "guess" $Y(t) = A\cos(t) + B\sin(t)$ even though there is no $\sin(t)$ in $f(t)$

because $Y' = -A\sin(t) + B\cos(t)$

$Y'' = -A\cos(t) - B\sin(t)$ <u>form</u> is kept!

→ ALWAYS include BOTH cosine and sine if $f(t)$ has either or both

**example** $y'' - y' - 2y = \sin(t)$

$$y = c_1 e^{-t} + c_2 e^{2t} + Y$$

$f(t)$ has $\sin(t)$, so $Y$ must contain BOTH $\cos(t)$ and $\sin(t)$

$$\left.\begin{aligned} Y &= A\cos(t) + B\sin(t) \\ Y' &= -A\sin(t) + B\cos(t) \\ Y'' &= -A\cos(t) - B\sin(t) \end{aligned}\right\} \text{ sub into } y'' - \cancel{2}y' - 2y = \sin(t)$$

$$-A\cos(t) - B\sin(t) + \cancel{2}A\sin(t) - \cancel{2}B\cos(t) - 2A\cos(t) - 2B\sin(t) = \sin(t)$$

$$(-3A - B)\cos(t) + (A - 3B)\sin(t) = \sin(t) + 0\cos(t)$$

$-3A - B = 0 \rightarrow B = -3A$

$A - 3B = 1 \qquad A + 9A = 1 \qquad A = \frac{1}{10}$

$B = -\frac{3}{10}$

$$\boxed{y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t)}$$

example $\quad y'' - y' - 2y = \underbrace{te^{3t}}_{\text{1st-deg polynomial times exponential}} + \underbrace{e^{t}\cos(3t)}_{\text{exponential time cosine or sine}}$

$$Y_1 = (At+B)e^{3t} = Ate^{3t} + Be^{3t} \quad \text{from } te^{3t}$$

$$Y_2 = (C\cos(3t) + D\sin(3t))e^{t}$$

$$Y = Ate^{3t} + Be^{3t} + Ce^{t}\cos(3t) + De^{t}\sin(3t)$$

$$Y' = \ldots$$

$$Y'' = \ldots.$$

plug into diff. eq

$\vdots$

$$y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{4}te^{3t} - \frac{5}{16}e^{3t} - \frac{11}{130}e^{t}\cos(3t) + \frac{3}{130}e^{t}\sin(3t)$$

this method has one serious complication

$$y'' - y' - 2y = e^{2t}$$

$$y = c_1 e^{-t} + \underline{\underline{c_2 e^{2t}}} + Y$$

$Y$ matches form of $f(t)$

$$Y = \underline{\underline{A e^{2t}}}$$

matches a fundamental solution

$$Y = ?$$