3.5 Undetermined Coeff. (continued)

$$ay'' + by' + cy = f(t)$$

$$y(t) = \underbrace{c_1 y_1 + c_2 y_2}_{\text{complementary solution from } ay''+by'+cy=0} + Y(t)$$

particular solution due to $f(t)$

undetermined coeff: $Y(t)$ resembles $f(t)$

polynomial $f(t) \to Y(t)$ is polynomial

exponential $f(t) \to Y(t)$ is exponential

cosine or sine $f(t) \to Y(t)$ has BOTH cosine and sine

what happens if $f(t)$ copies one of the fundamental solutions?

for example, $y'' + y' - 2y = e^t$

$y'' + y' - 2y = 0 \qquad r^2 + r - 2 = 0 \qquad (r+2)(r-1) = 0$

$r = -2, \; r = 1$

$y = c_1 e^{-2t} + c_2 e^t + Y$

$f(t)$ is e^t, so normally we say $Y = Ae^t$

$Y' = Ae^t$

$Y'' = Ae^t$

Sub into $y'' + y' - 2y = e^t$

$$Ae^t + Ae^t - 2Ae^t = e^t$$

$$0 = e^t \quad \text{not true!}$$

this says $Y = Ae^t$ is NOT right

because $f(t)$ copies y_1 or y_2

let's use the reduction of order to see what Y should look like

$$y'' + y' - 2y = e^t$$

$$y_1 = e^{-2t}, \quad y_2 = e^t$$

reduction of order: if we have one solution, we can use it to find the rest

$$y = v(t)\,y_1 \quad \text{or} \quad y = v(t)\,y_2$$

here, let's use $y_2 = e^t$

$$y = ve^t \qquad \text{find } v$$

$$y' = ve^t + v'e^t$$

$$y'' = ve^t + v'e^t + v'e^t + v''e^t = ve^t + 2v'e^t + v''e^t$$

sub into $y'' + y' - 2y = e^t$

$$\cancel{v}e^t + 2v'e^t + v''e^t + \cancel{v}e^t + v'e^t - \cancel{2v}e^t = e^t$$

divide by e^t

$$v'' + 3v' = 1$$

let $w = v'$

$$w' + 3w = 1 \qquad \text{linear 1st-order}$$

$$\mu = e^{\int 3\,dt} = e^{3t}$$

$$\frac{d}{dt}(e^{3t}w) = e^{3t}$$

$$e^{3t}w = \frac{1}{3}e^{3t} + a$$

$$v' = w = \frac{1}{3} + ae^{-3t}$$

$$v = \frac{1}{3}t + -\frac{1}{3}ae^{-3t} + b \qquad y = ve^t$$

$$y = -\frac{1}{3}ae^{-3t}\cdot e^t + b\cdot e^t + \frac{1}{3}t\cdot e^t$$

$$\boxed{y = \underbrace{c_1e^{-2t} + c_2e^t}_{\text{complementary}} + \underbrace{\frac{1}{3}te^t}_{\text{particular form is } te^t}}$$

attach an extra t if $f(t)$ copies y_1 or y_2

example

$$y'' - y' = 3$$

$$y = c_1 y_1 + c_2 y_2 + Y$$

$y'' - y' = 0$ $\quad r^2 - r = 0$ $\quad r = 0,\ r = 1$

$y_1 = e^{0t} = 1$ $\quad y_2 = e^t$

$$y = c_1 + c_2 e^t + Y$$

$f(t) = 3$ constant

so, normally we say $Y = A$ but $A = A \cdot 1$ which copies $c_1 = c_1 \cdot 1$

attach t to the Y that copies y_1 or y_2

$Y = At$

$Y' = A$

$Y'' = 0$

$-A = 3$ $\quad A = -3$

so, $\boxed{y = c_1 + c_2 e^t - 3t}$

example $y'' - y' = t$

$y = c_1 + c_2 e^t + Y$

$f(t) = t$ is 1st-order polynomial

guess the form: $Y = At + B$

copies c_1
give it a t
but it will copy At
so we give BOTH a t

fix: $Y = At^2 + Bt$

or $Y = t(At + B)$

example $y'' - y' = t + e^{2t}$

$y = c_1 + c_2 e^t + Y$

$Y = \underbrace{At + B}_{\text{extra } t \text{ for copying } y_1 \text{ or } y_2} + Ce^{2t}$

no t because e^{2t} is not copying anything

fixed Y is

$Y = t(At + B) + Ce^{2t}$

<u>example</u> $y'' + y = t\cos(t)$

$y'' + y = 0 \quad r^2 + 1 = 0 \quad r = \pm i$

$y = c_1 \cos(t) + c_2 \sin(t) + Y$

$f(t) = t\cos(t)$ cosine or sine → cosine AND sine in Y

1st-degree polynomial

tentative Y: $Y = (At + \underbrace{B)\cos(t)}_{\text{copying}} + (Ct + \underbrace{D)\sin(t)}_{\text{copying}}$

copying $c_1\cos(t)$ but getting t copies $At\cos(t)$ so, BOTH $At+B$ get extra t

copying $c_2\sin(t)$

same story here

corrected Y: $Y = t(At+B)\cos(t) + t(Ct+D)\sin(t)$

$Y' = \ldots$

$Y'' = \ldots$

example $\underbrace{y''+2y'+5y}_{r^2+2r+5=0} = 3te^{-t}\cos(2t) - 2te^{-2t}\cos(t)$

$$r^2+2r+5=0 \qquad r = \frac{-2 \pm \sqrt{4-4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t) + Y$$

tentative Y:

$$Y = t\left[(At+B)e^{-t}\cos(2t) + (Ct+D)e^{-t}\sin(2t)\right]$$

$$+ \underbrace{(Et+F)e^{-2t}\cos(t) + (Gt+H)e^{-2t}\sin(t)}_{\text{safe. NO copying}}$$

t for all in first two big terms