3.6 Variation of Parameters

another way to solve $y'' + p(t)y' + q(t)y = g(t)$ nonhomogeneous

solution: $y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{complementary}} + Y$ particular

solution to $y'' + p(t)y' + q(t)y = 0$

undetermined coeff: Y resembles $g(t)$

Y keeps its form after differentiation

$y'' + y' = \sec(t)$ $Y = ?$

can't handle this with undetermined coeff!

if we tried $Y = A\sec(t)$

$Y' = A\underbrace{\sec(t)\tan(t)}_{\text{changed form!}}$

what if $Y = A\sec(t) + B\tan(t)$

$Y' = A\sec(t)\tan(t) + B\sec^2(t)$

variation of parameters: solution is of the form

$$\boxed{y = \underset{\text{"parameters"}}{u_1(t)}\, \underset{\text{complementary}}{y_1} + u_2(t)\, y_2}$$

if we can find $u_1(t)$ and $u_2(t)$, we can separate out the particular solution Y.

$$y'' + p(t)y' + q(t)y = g(t)$$

solution: $y = u_1(t)y_1 + u_2(t)y_2$ sub into

$$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y'' = u_1 y_1'' + u_1' y_1' + u_1' y_1' + u_1'' y_1 + u_2 y_2'' + u_2' y_2' + u_2' y_2' + u_2'' y_2$$

if we sub these into $y'' + py' + qy = g$, eventually we will have <u>TWO</u> <u>2nd-order</u> eqs. for u_1 and u_2

TWO unknowns (u_1, u_2), one eq: $y'' + py' + qy = g$

$\longrightarrow$ we can impose <u>one</u> condition so, we will impose the condition that eliminates the 2nd-order eqs. (so just u_1' and u_2')

$$\rightarrow \boxed{u_1'y_1 + u_2'y_2 = 0}$$

update the eqs:

$$\left.\begin{aligned} y &= u_1y_1 + u_2y_2 \\ y' &= u_1y_1' + u_2y_2' \\ y'' &= u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2' \end{aligned}\right\} \text{sub into } y'' + py' + qy = g$$

$$u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2' + pu_1y_1' + pu_2y_2' + qu_1y_1 + qu_2y_2 = g$$

$$u_1\underbrace{(y_1'' + py_1' + qy_1)}_{0} + u_2\underbrace{(y_2'' + py_2' + qy_2)}_{0} + u_1'y_1' + u_2'y_2' = g$$

because y_1 is a complementary solution ($y'' + py' + qy = 0$)

same reason

so, $\boxed{u_1'y_1' + u_2'y_2' = g}$

solve the two boxed eqs
for u_1' and u_2'
then integrate to find u_1, u_2

example $y'' + y = \sec(t)$

complementary: $y'' + y = 0 \quad r^2 + 1 = 0 \quad r = \pm i$

$$y_1 = \cos(t)$$
$$y_2 = \sin(t)$$

$$y = u_1 y_1 + u_2 y_2$$

solve $\quad u_1' y_1 + u_2' y_2 = 0$

$$u_1' y_1' + u_2' y_2' = g(t) = \sec(t)$$

$$u_1' \cos(t) + u_2' \sin(t) = 0$$

$$-u_1' \sin(t) + u_2' \cos(t) = \sec(t)$$

$u_1' = -\dfrac{\sin(t)}{\cos(t)} u_2'$ sub into 2nd eq.

$u_2' \dfrac{\sin^2(t)}{\cos(t)} + u_2' \cos(t) = \sec(t)$ mult. by $\cos(t)$

$$u_2' \left[\sin^2(t) + \cos^2(t)\right] = 1$$

$$u_2' = 1 \longrightarrow \boxed{u_2 = t + C_2}$$

$$u_1' = -\frac{\sin(t)}{\cos(t)} u_2'$$

$$u_1' = -\frac{\sin(t)}{\cos(t)} \longrightarrow \boxed{u_1 = \ln|\cos(t)| + C_1}$$

general solution to $y'' + y = \sec(t)$ is then

$$y = u_1 y_1 + u_2 y_2$$

$$= \left(\ln|\cos(t)| + C_1\right)\cos(t) + (t + C_2)\sin(t)$$

$$= \underbrace{C_1\cos(t) + C_2\sin(t)}_{\text{complementary}} + \underbrace{\cos(t)\ln|\cos(t)| + t\sin(t)}_{\text{particular } (Y)}$$

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Wronskian

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2 g \\ y_1 g \end{bmatrix}$$

$$\boxed{u_1' = -\frac{y_2 g}{W}}$$

$$\boxed{u_2' = \frac{y_1 g}{W}}$$

undetermined coeff : multiply by t if ~~partic~~ right side copies y_1 or y_2

variation of parameters takes care of that <u>automatically</u>