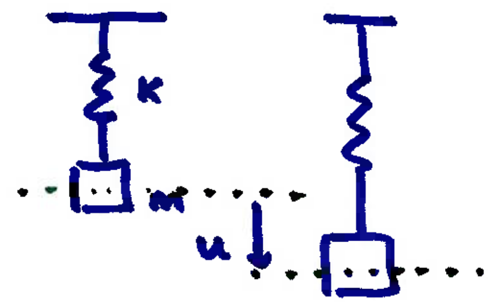
3.7 (continued)



u: elogation from equilibrium

u > 0 down

last time: $mu'' + ku = 0$

example $m = \frac{5}{16}$ $k = 20$

$u(0) = \frac{1}{6}$, $u'(0) = -1$ ← upward

solved $u(t) = \frac{1}{6}\cos(8t) - \frac{1}{8}\sin(8t)$

want to turn $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

into $u(t) = R\cos(\omega_0 t - \delta)$

identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$u(t) = R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t)$

we see: $A = R\cos(\delta)$ $B = \sin(\delta)$

$A^2 + B^2 = R^2\cos^2(\delta) + R^2\sin^2(\delta)$

$A^2 + B^2 = R^2$

$$\boxed{R = \sqrt{A^2 + B^2}}$$

$$\boxed{\tan(\delta) = \frac{B}{A}}$$

$$u(t) = \frac{1}{6}\cos(8t) - \frac{1}{8}\sin(8t)$$

$$R = \sqrt{\left(\tfrac{1}{6}\right)^2 + \left(-\tfrac{1}{8}\right)^2} = \frac{5}{24}$$

$$\tan(\delta) = \frac{-1/8}{1/6} = -\frac{3}{4}$$

$$\delta = \tan^{-1}\left(-\tfrac{3}{4}\right) \approx -0.64 \text{ rad}$$

$$\boxed{u(t) = \frac{5}{24}\cos(8t - (-0.64))}$$

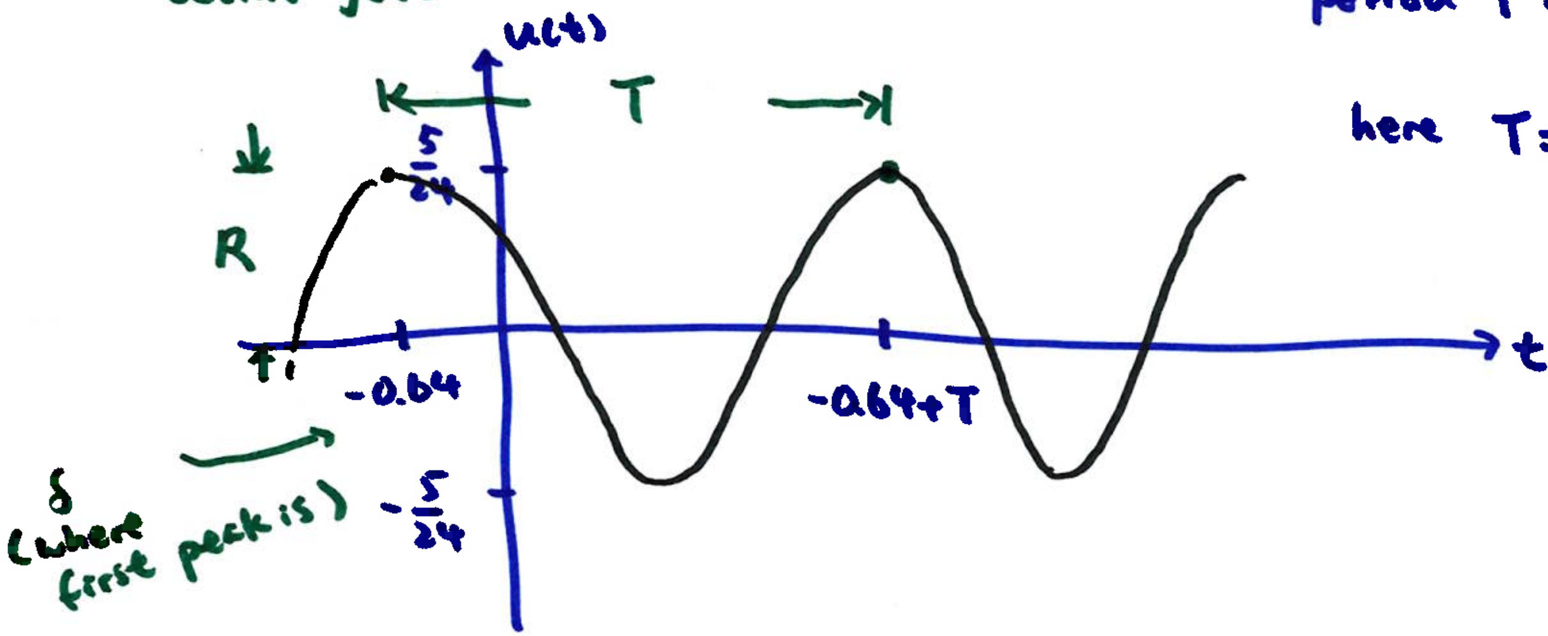
phase shift (δ)
how the peak of cosine is moved ($\delta > 0 \rightarrow$ move RIGHT)

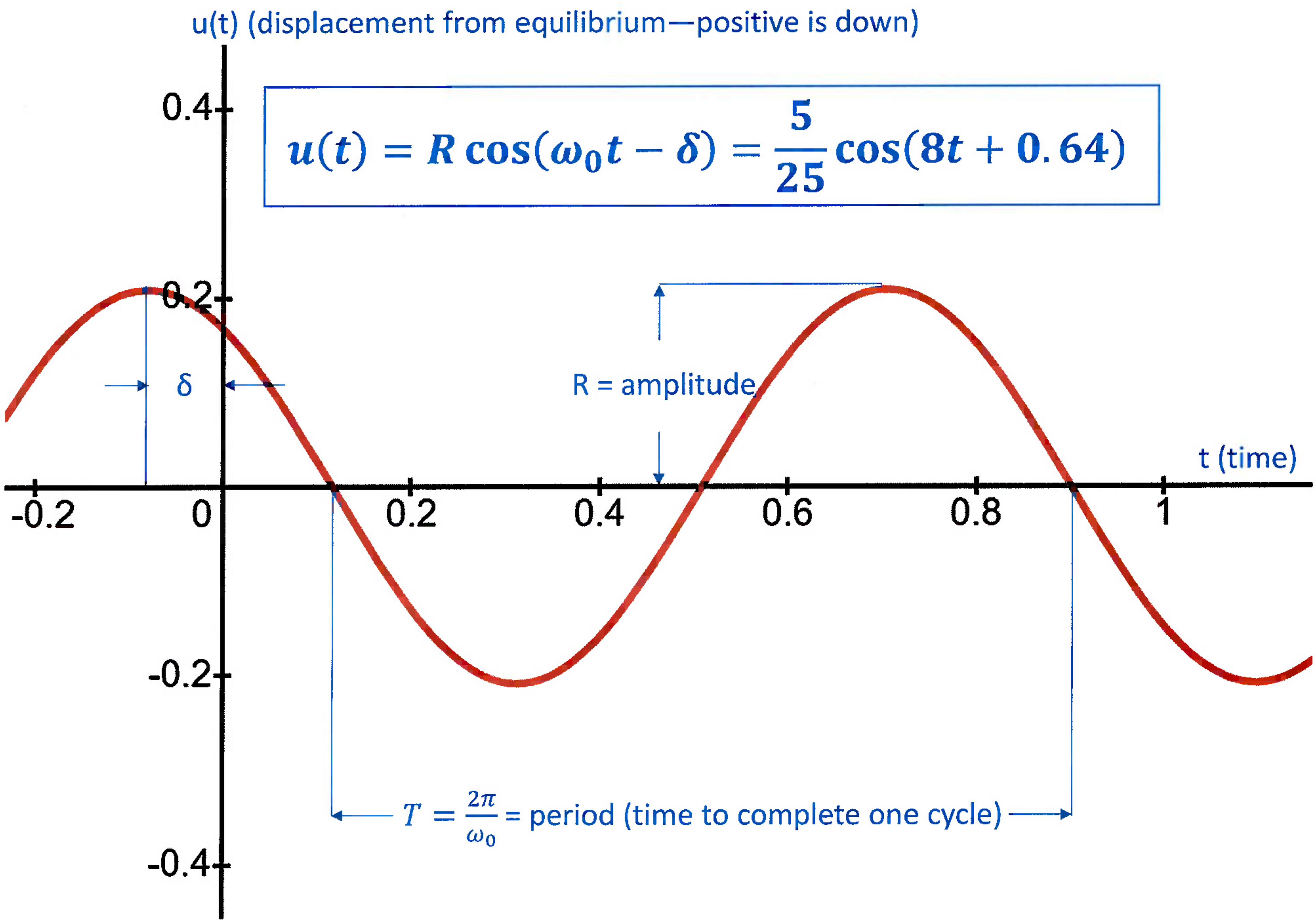
amplitude (R)
how far up and down cosine goes

angular frequency (ω_0)
how fast the oscillation is

period $T = \frac{2\pi}{\omega_0}$

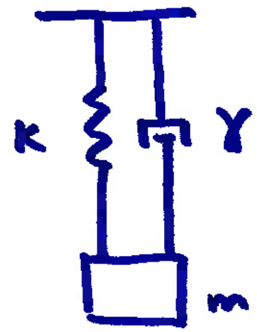
here $T = \frac{2\pi}{8} = \frac{\pi}{4}$





u(t) (displacement from equilibrium—positive is down)
$u(t) = R\cos(\omega_0 t - \delta) = \frac{5}{25}\cos(8t + 0.64)$
0.4
0.2
-0.2
-0.4
δ
R = amplitude
t (time)
-0.2
0
0.2
0.4
0.6
0.8
1
$T = \frac{2\pi}{\omega_0}$ = period (time to complete one cycle)

now let's add a damper



$(k>0, \gamma>0, m>0)$

γ: damping constant

resists velocity (u') in the opposite direction

force $= -\gamma u'$

opposite direction

just like spring $-ku$

new equation $\boxed{mu'' + \gamma u' + ku = 0}$ $u(0) = u_0$, $u'(0) = v_0$

$(u>0$ down$)$

characteristic eq: $mr^2 + \gamma r + k = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

$\gamma^2 - 4km$ determines solution type if $\gamma^2 - 4km < 0$ → r's are complex

solutions are decaying cosines and sines → has oscillations

$\boxed{\text{underdamped}}$ "weak" damping

if $\gamma^2 - 4km = 0$ → r's are real and repeated

solutions are decaying exponentials No oscillations

$\boxed{\text{critically damped}}$ damped "just right"

if $\gamma^2 - 4km > 0$ → r's are real, distinct, negative

solutions are also decaying exponentials

No oscillations

$\boxed{\text{overdamped}}$ "strong" damping back the example we did last time, add a damper

$$mu'' + \gamma u' + ku = 0 \qquad m = \frac{5}{16}, \quad k = 20, \quad \gamma = 4 \qquad \gamma^2 < 4km$$

$$u(0) = \frac{1}{6}, \quad u'(0) = -1$$

underdamped

$$\frac{5}{16}u'' + 4u' + 20u = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot \frac{5}{16} \cdot 20}}{2\left(\frac{5}{16}\right)} = \frac{-4 \pm \sqrt{-9}}{\frac{5}{8}} = -\frac{32}{5} \pm \frac{24}{5} i$$

$$u(t) = C_1 e^{-\frac{32}{5}t} \cos\left(\frac{24}{5}t\right) + C_2 e^{-\frac{32}{5}t} \sin\left(\frac{24}{5}t\right)$$

using $u(0) = \frac{1}{6}$, $u'(0) = -1$, we get

$$u(t) = \frac{1}{6} e^{-\frac{32}{5}t} \cos\left(\frac{24}{5}t\right) + \frac{1}{72} e^{-\frac{32}{5}t} \sin\left(\frac{24}{5}\right)t$$

$$= e^{-\frac{32}{5}t}\left[\underbrace{\frac{1}{6}\cos\left(\frac{24}{5}t\right) + \frac{1}{72}\sin\left(\frac{24}{5}\right)t}_{\text{turn into } R\cos(\omega t - \delta)}\right]$$

$$= e^{-\frac{32}{5}t}\left[\frac{\sqrt{145}}{72}\cos\left(\frac{24}{5}t - \tan^{-1}\left(\frac{1}{12}\right)\right)\right]$$

phase shift

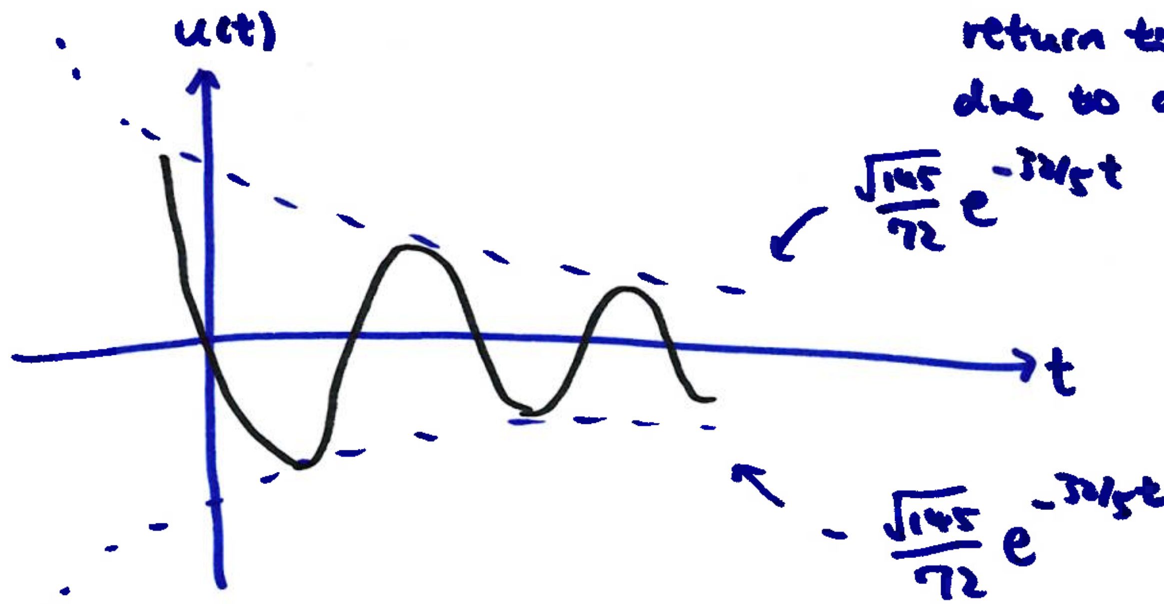
$$= \boxed{\underbrace{\frac{\sqrt{145}}{72}e^{-\frac{32}{5}t}}_{\text{amplitude}}\left[\cos\left(\frac{24}{5}t - 0.08\right)\right]}$$

amplitude goes to 0 as $t \to \infty$

quasi frequency (damped freq)

quasi period $T = \frac{2\pi}{\omega}$

"quasi" because we never return to the same u due to damping

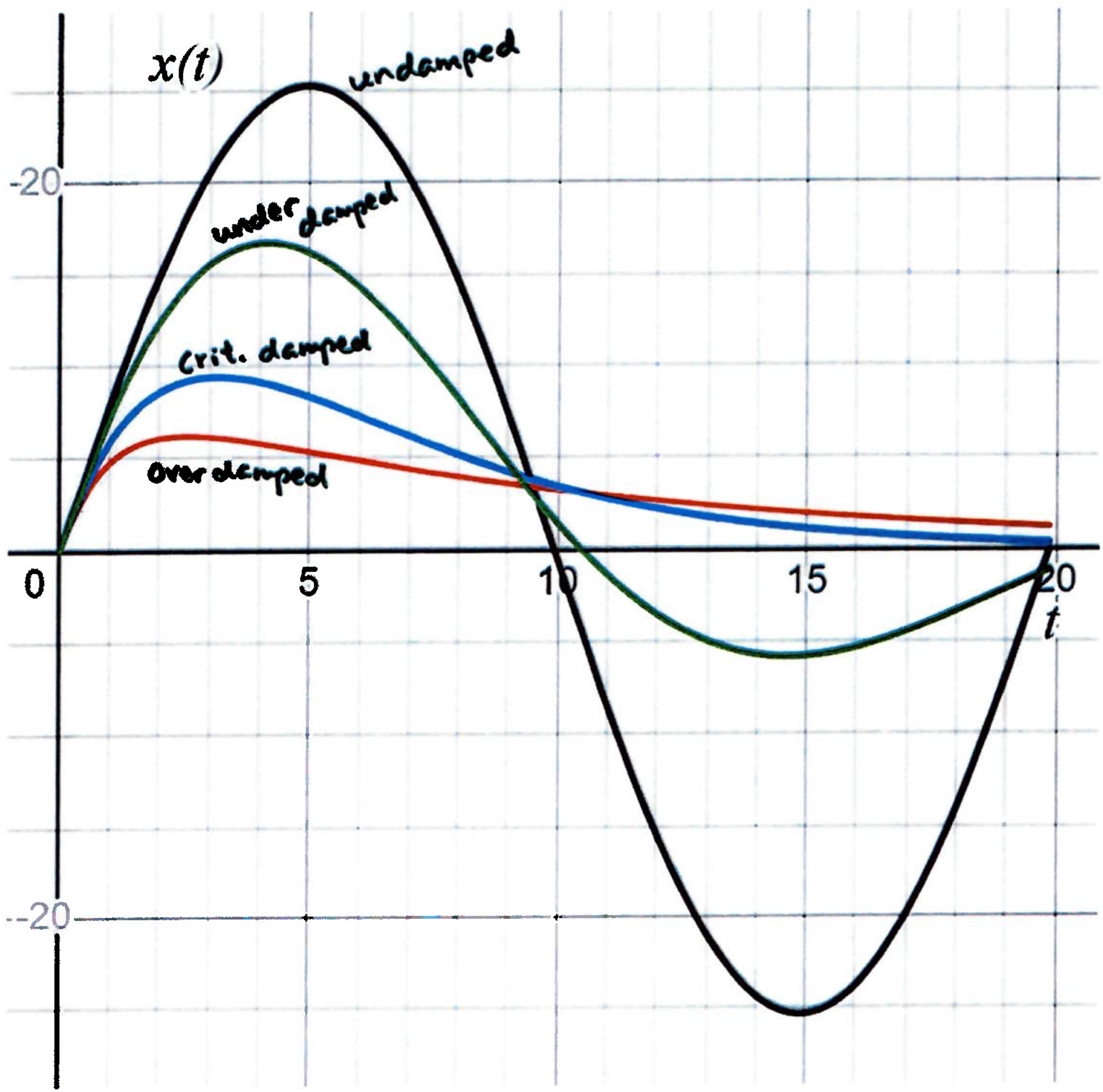


undamped: $\frac{5}{24} \cos(8t + 0.64)$

damped: $\frac{\sqrt{145}}{72} e^{-32/5\, t} \left(\cos \frac{24}{5}t - 0.08 \right)$

frequency: $8 \to \frac{24}{5}$ damper <u>decreased</u> the frequency

period: $\frac{2\pi}{8} \to \frac{2\pi}{24/5}$ period <u>increased</u>



x(t)
undamped
under damped
crit. damped
over damped
20
-20
0
5
10
15
20
t