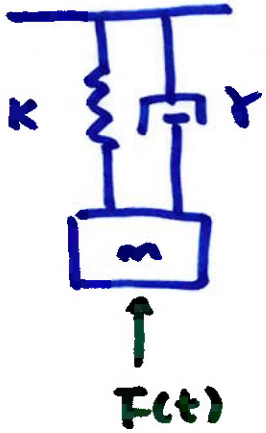
3.8 Forced Periodic Vibrations



$$mu'' + \gamma u' + ku = F(t)$$

let's focus on a periodic $F(t)$

$$F(t) = F_0 \cos(\omega t)$$

F_0: magnitude

ω: freq. of applied force

let's start w/ undamped case: $\gamma = 0$

$$\underbrace{mu'' + ku}\ = F_0 \cos(\omega t)$$

characteristic eq $\quad mr^2 + k = 0 \qquad r = \pm\sqrt{\frac{k}{m}}\, i$

$$u(t) = C_1 \cos\Big(\underbrace{\sqrt{\tfrac{k}{m}}}\, t\Big) + C_2 \sin\Big(\underbrace{\sqrt{\tfrac{k}{m}}}\, t\Big) + U$$

U ← particular solution due to $F(t)$

natural freq. of mass-spring system $\quad \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

particular solution $\quad U = A\cos(\omega t) + B\sin(\omega t)$ $\quad \boxed{\omega \neq \omega_0}$

$U' = \dots$

$U'' = \dots$

sub into diff. eq.

finding: $B = 0$, $\quad A = \dfrac{F_0}{k - m\omega^2} = \dfrac{F_0}{m\left(\frac{k}{m} - \omega^2\right)} = \dfrac{F_0/m}{(\omega_0^2 - \omega^2)}$

general solution:

$$\boxed{u(t) = \underbrace{c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)}_{\substack{\text{freq. } \omega_0 \\ \text{period } \frac{2\pi}{\omega_0}}} + \underbrace{\frac{F_0/m}{(\omega_0^2 - \omega^2)}\cos(\omega t)}_{\substack{\text{freq. } \omega \\ \text{period } \omega}}}$$

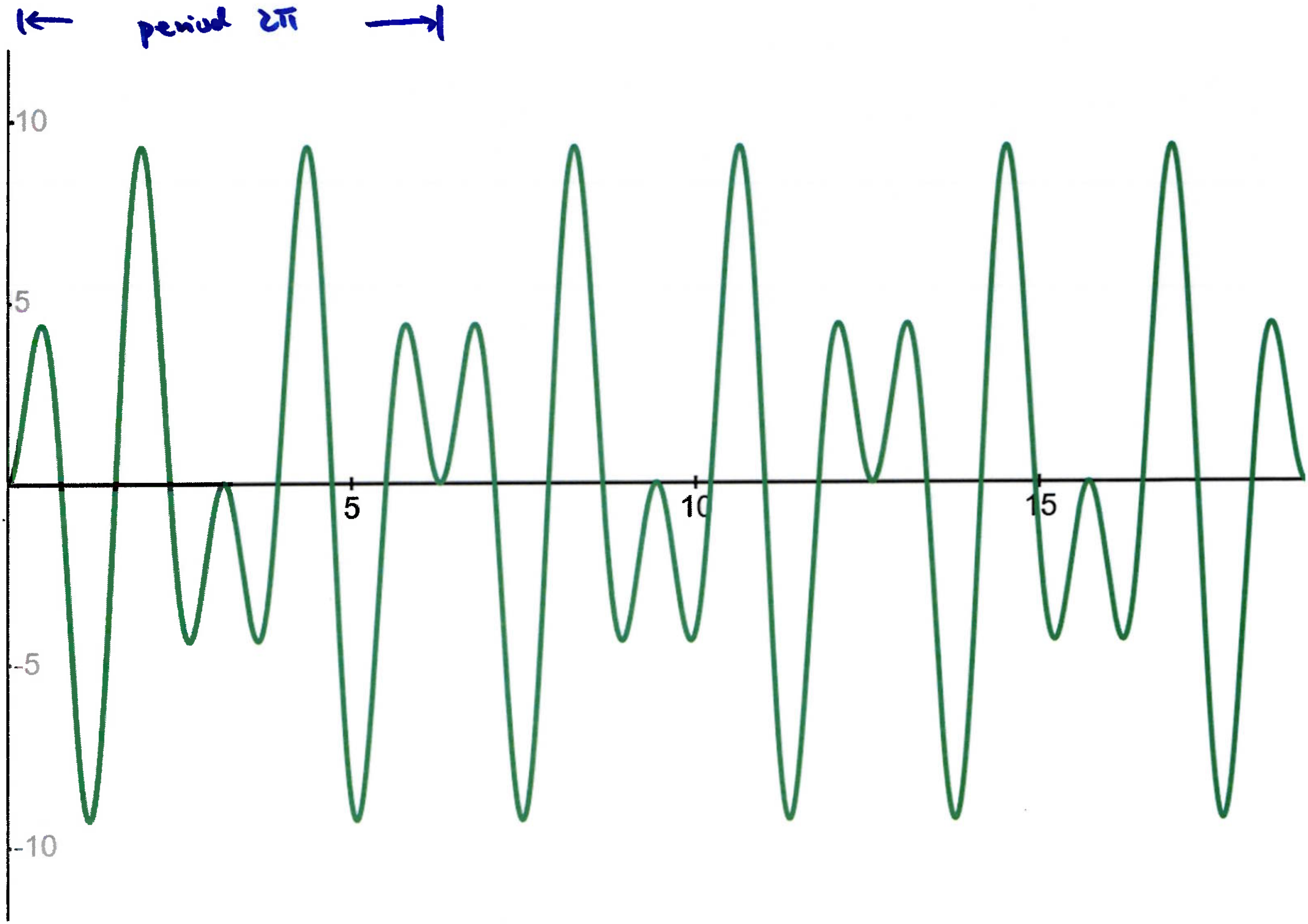
as an example, if $m = 1$, $k = 9$, $F_0 = 80$, $\omega = 5$, $u(0) = u'(0) = 0$

$$u(t) = \underbrace{5\cos(3t)}_{\text{mass-spring}} - \underbrace{5\cos(5t)}_{\text{from external force}}$$

1st term period $\frac{2\pi}{3}$

2nd term " $\frac{2\pi}{5}$

ratio is rational so overall period is the least common multiple


period 2π
10
5
-5
-10
5
10
15

interesting thing happens if $\omega \approx \omega_0$

for example, $m = 0.1$, $F_0 = 50$, $k = 302.5$, $\omega = 45$

$$\omega_0 = \sqrt{\frac{k}{m}} = 55$$

$$u(t) = C_1 \cos(55t) + C_2 \sin(55t) + \frac{1}{2}\cos(45t)$$

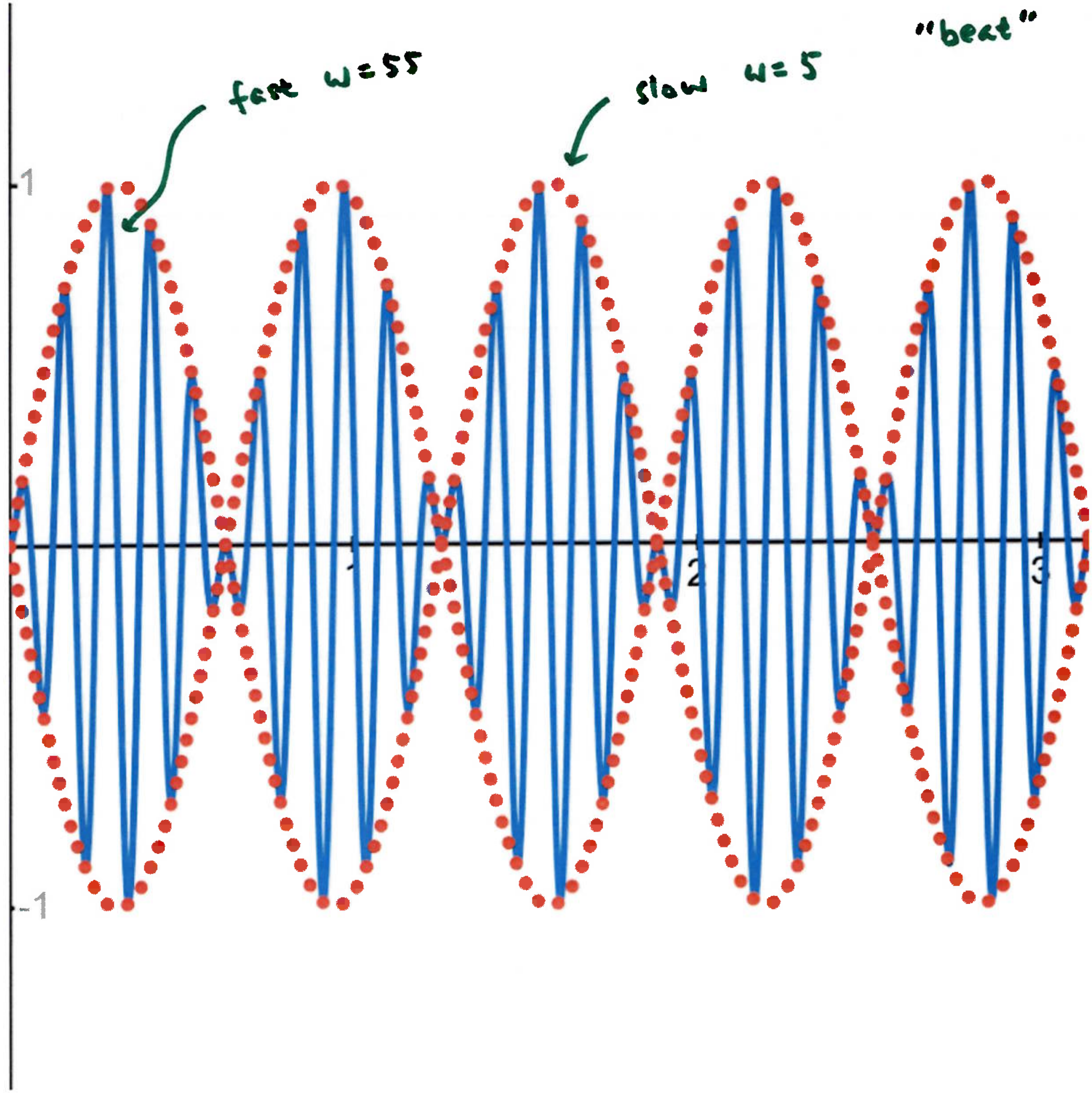
if $u(0) = u'(0) = 0$

$$u(t) = -\frac{1}{2}\cos(55t) + \frac{1}{2}\cos(45t)$$

if use the identity $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$A = \frac{1}{2}(55+45)$ $\quad B = \frac{1}{2}(55-45)$

$$u(t) = \underbrace{\sin(5t)}_{\substack{\text{slow} \\ \text{low } \omega}} \underbrace{\sin(50t)}_{\substack{\text{fast} \\ \text{high } \omega}}$$



"beat"
fast w=55
slow w=5
1
-1
1
2
3

what if $\omega_0 = \omega$?

$U = A\cos(\omega t) + B\sin(\omega t)$ needs adjusted

$= A\,\underline{\underline{t}}\cos(\omega t) + B\,\underline{\underline{t}}\sin(\omega t)$

if $m=1$, $F_0 = 100$, $\omega_0 = 50$, $\omega = 50$ $u(0) = u'(0) = 0$

$$mu'' + ku = F_0\cos(\omega t)$$

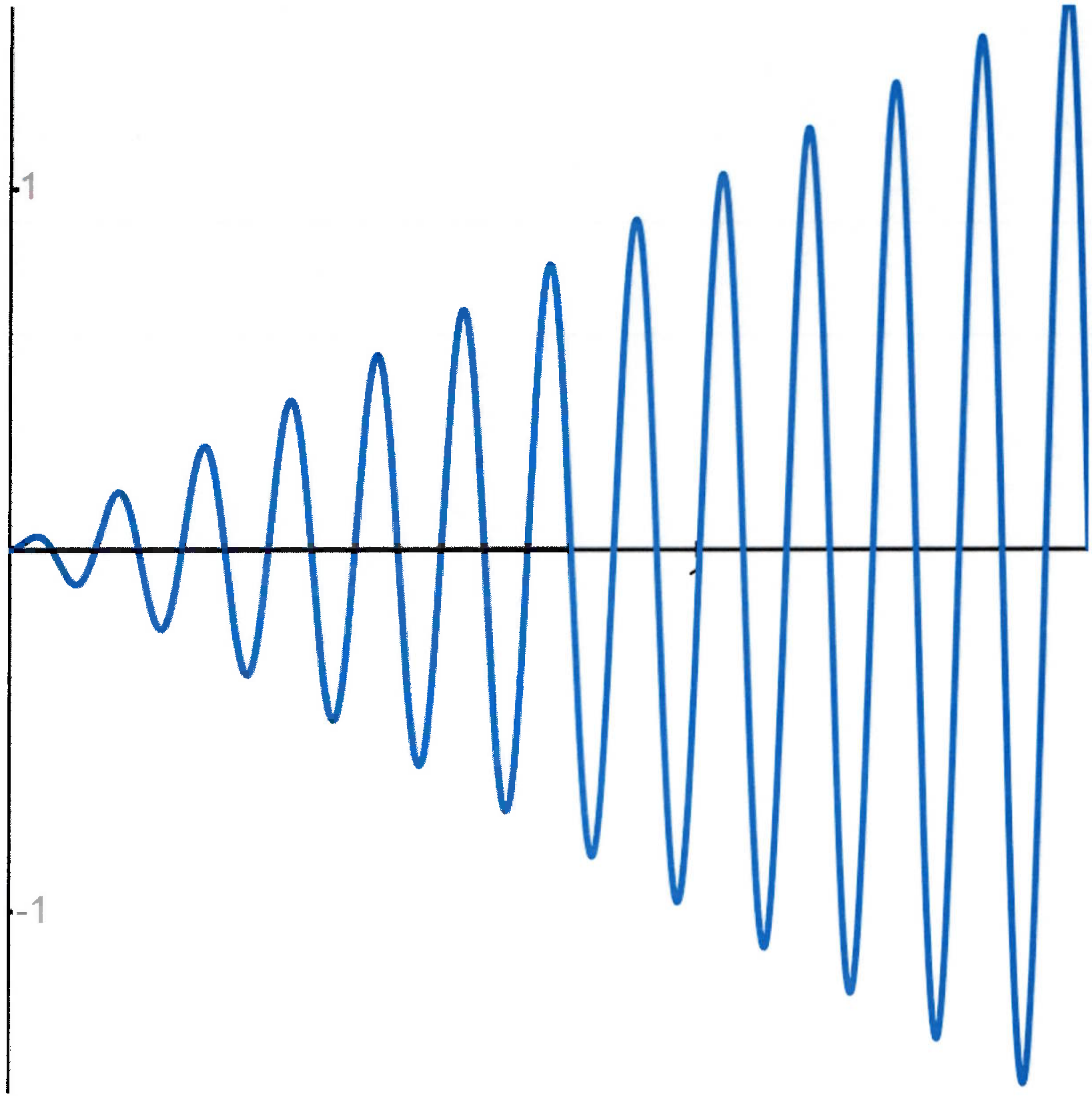
has solution $u(t) = \underline{\underline{t}}\sin(50t)$

makes $u \to \infty$ as $t \to \infty$

displacement $\to \infty$

this is called <u>resonance</u>

BAD for structures



1
-1

now let's put the damper back in

$$\underbrace{mu'' + \gamma u' + ku}_{r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}} = F_0 \cos(\omega t)$$

$$u(t) = \underbrace{c_1 u_1 + c_2 u_2}_{\text{depending on } r} + U$$

$U = A\cos(\omega t) + B\sin(\omega t)$

$U' = \ldots$

$U'' = \ldots$

sub into $mu'' + \gamma u' + ku = F_0 \cos(\omega t)$

$$A = \frac{(k - m\omega^2) F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2} \qquad B = \frac{\gamma \omega F_0}{(k - m\omega^2)^2 + (\gamma\omega)^2}$$

as long as $\gamma \neq 0$, both A and B are bounded so, u will NOT go to ∞ as $t \to \infty$

$m=1$, $k=26$, $\gamma=2$, $F_0=82$, $\omega=4$ $\quad u(0)=6$, $u'(0)=0$

$k=26$: Spring fights back with force 26 N/m

$\gamma=2$: damper fights back with force of 2 N/m/s

$$u''+2u'+26u=82\cos(4t)$$

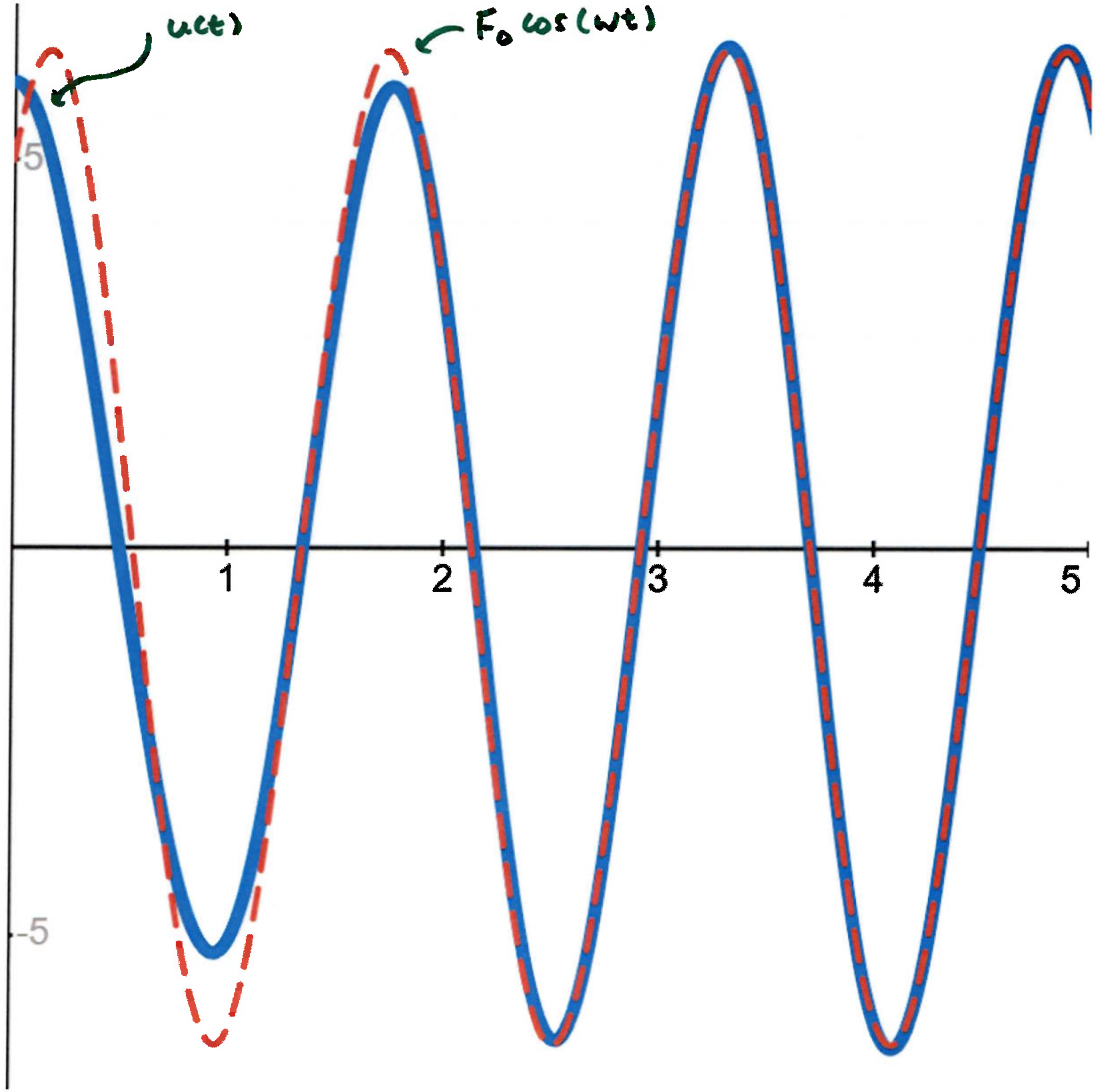
$\vdots$

$$u(t)=\underbrace{\overbrace{e^{-t}(\cos 5t-3\sin 5t)}^{\text{complementary}}}_{\text{transient solution}}+\underbrace{\overbrace{5\cos 4t-4\sin 4t}^{\text{due to input}}}_{\text{steady-state solution}}$$

transient solution: goes to 0 as $t\to\infty$ due to the damper which supplies e to negative #

steady-state solution (or forced response): never goes away

$\lim_{t\to\infty} u(t) \approx F_0\cos(\omega t)$ but with a phase shift



u(t)
$F_0\cos(\omega t)$
5
-5
1
2
3
4
5