4.1 - 4.2 Higher-Order ODEs

most of what we know about 2nd-order extend to higher-order ODEs.

n^{th}-order linear

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

In the form with leading coefficient 1

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-2}} + p_2(t)\frac{d^{n-2}y}{dt^{n-2}} + \cdots + p_n(t)y = g(t)$$

the equation has a unique solution on some interval I where

all $p_1(t), p_2(t), \ldots, p_n(t), g(t)$ are continuous and the interval

I contains the intial t (t_0).

up to y'

2nd-order : 2 initial conditions, usually $y(t_0), y'(t_0)$

n^{th}-order : n initial conditions

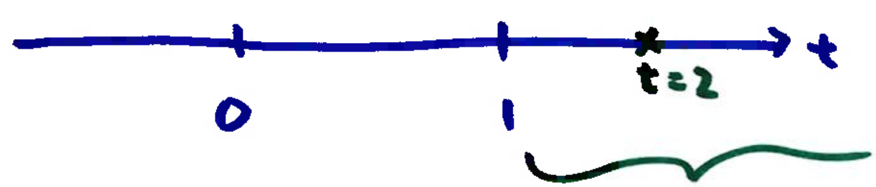
$$\left.\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \\ y''(t_0) &= y_0'' \\ &\vdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}\right\} \text{ n of these}$$

for example, $t(t-1)\,y^{(4)} + e^t y'' + 4t^2 y = 0$

$y(2) = 1, \quad y'(2) = 0, \quad y''(2) = 3, \quad y'''(2) = -1$

$$y^{(4)} + \boxed{\frac{e^t}{t(t-1)}}\, y'' + \boxed{\frac{4t^2}{t(t-1)}}\, y = 0$$

continuous $t \neq 0,\ t \neq 1$



$I : (1, \infty)$

n^{th}-order $\rightarrow$ n fundamental solutions

$y_1, y_2, y_3, \ldots, y_n$

general solution: $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n$

$c_1, c_2, \ldots, c_n$ come from the n initial conditions

given some initial conditions, can we always find the constants?

suppose for a 4^{th}-order eq. we have y_1, y_2, y_3, y_4

initial conditions y_0, y_0', y_0'', y_0'''

general solution: $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$

at t_0,

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 = y_0$$

$$c_1 y_1' + c_2 y_2' + c_3 y_3' + c_4 y_4' = y_0'$$

$$c_1 y_1'' + c_2 y_2'' + c_3 y_3'' + c_4 y_4'' = y_0''$$

$$c_1 y_1''' + c_2 y_2''' + c_3 y_3''' + c_4 y_4''' = y_0'''$$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \\ y_0'' \\ y_0''' \end{bmatrix}$$

a unique solution for c_1, c_2, c_3, c_4 if

$$\underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{\text{Wronskian } W[y_1, y_2, y_3, y_4]} \neq 0$$

$W \neq 0 \rightarrow$ the solutions are <u>linearly independent</u>

for example, $y''' + 2y'' - y' - 2y = 0$ has solutions e^{t}, e^{-t}, e^{-2t}

$$W = \begin{vmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & -e^{-t} & -2e^{-2t} \\ e^{t} & e^{-t} & 4e^{-2t} \end{vmatrix}$$

cofactor expansion

pick any row or column to expand, here, we pick row 1

$$= e^{t} \begin{vmatrix} -e^{-t} & -2e^{-2t} \\ e^{-t} & 4e^{-2t} \end{vmatrix} - e^{-t} \begin{vmatrix} e^{t} & -2e^{-2t} \\ e^{t} & 4e^{-2t} \end{vmatrix} + e^{-2t} \begin{vmatrix} e^{t} & -e^{-t} \\ e^{t} & e^{-t} \end{vmatrix}$$

$$= e^{t}(-4e^{-3t} + 2e^{-3t}) - e^{-t}(4e^{-t} + 2e^{-t}) + e^{-2t}(e^{0} + e^{0})$$

$$= -2e^{-2t} - 6e^{-2t} + 2e^{-2t} = -6e^{-2t} \neq 0 \text{ for any } t$$

if W is not identically zero (not zero all the time) on some interval I, then ~~the~~ a unique solution exists throughout I and the fundamental solutions are linearly independent

constant-coefficient eqs: $a_1 y^{(n)} + a_2 y^{(n-1)} + \dots + a_n y = 0$
homogeneous

just like for 2nd-order, the solutions are of the form e^{rt}

characteristic eq: $\underbrace{a_1 r^n + a_2 r^{n-1} + \dots + a_n = 0}$

n roots $\rightarrow$ n e^{rt} solutions

for example, $y''' - 3y'' + 3y' - y = 0$

char. eq: $r^3 - 3r^2 + 3r \underbrace{- 1} = 0$ cubic

$\hookrightarrow$ one root is a factor of this

-1 has factors $\underbrace{1, -1}$

one of those is a root

is $r=1$ a root?

$1 - 3 + 3 - 1 = 0$ true, so $r=1$ is a root

$$r^3-3r^2+3r-1 = (r-1)\underbrace{(ar^2+br+c)}_{?}$$

$$r^2-2r+1$$

polynomial long division

$$\begin{array}{r} r^2-2r+1 \\ r-1 \,\big)\, \overline{r^3-3r^2+3r-1} \\ -(r^3-r^2) \\ \hline -2r^2+3r \\ -(-2r^2+2r) \\ \hline r-1 \end{array}$$

$$(r-1)(r^2-2r+1)=0$$

$$(r-1)(r-1)(r-1)=0 \qquad r=1,1,1$$

same repeated roots rule

$$y_1=e^t,\quad y_2=te^t,\quad y_3=t^2e^t$$

$$y^{(6)} - y'' = 0$$

$$r^6 - r^2 = 0$$

$$r^2(r^4 - 1) = 0$$

$$r^2(r^2+1)(r^2-1) = 0$$

$$r = \underbrace{0, 0,}\ \underbrace{1, -1,}\ \underbrace{i, -i}$$

$y_1 = 1$ $\quad y_3 = e^t$ $\quad y_5 = \cos(t)$

$y_2 = t$ $\quad y_4 = e^{-t}$ $\quad y_6 = \sin(t)$

general solution

$$y = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 \cos(t) + c_6 \sin(t)$$