4.3 Undetermined Coefficients

idea is the same as for 2nd-order

particular solution resembles the nonhomogeneous term

throw "t" at terms that copy something.

<u>example</u> $y''' + y'' + y' + y = e^{-t} + 4t$

Solution: $y = y_c + Y$ ← particular (due to right side)

complementary (right side = 0)

$$r^3 + r^2 + r + 1 = 0$$

$$r^2(r+1) + (r+1) = 0$$

$$(r+1)(r^2+1) = 0 \qquad r = -1, \ \underbrace{r = -i, \ r = i}_{\text{complex ALWAYS in pairs}}$$

$$y = c_1 e^{-t} + c_2 \cos(t) + c_3 \sin(t) + Y$$

now Y: right side is $\underset{\substack{\text{exponential} \\ \text{guess } Ae^{-t}}}{e^{-t}} + \underset{\substack{\text{1st-degree} \\ \text{polynomial} \\ \text{guess } Bt+C}}{4t}$

initial Y: $Y = \underbrace{Ae^{-t}}_{\text{copying } c_1e^{-t}} + Bt + C$ fix: multiplying by t

fixed Y: $Y = Ate^{-t} + Bt + C$ sub into the diff. eq.

$$Y' = -Ate^{-t} + Ae^{-t} + B$$

$$Y'' = Ate^{-t} - 2Ae^{-t}$$

$$Y''' = -Ate^{-t} + 3Ae^{-t}$$

$$-\cancel{At}e^{-t} + 3Ae^{-t} + \cancel{At}e^{-t} - 2Ae^{-t} - \cancel{At}e^{-t} + Ae^{-t} + B + \cancel{At}e^{-t} + Bt + C = e^{-t} + 4t$$

$$2Ae^{-t} + Bt + (B+C) = e^{-t} + 4t$$

equate coefficients

$2A = 1$ $B = 4$ $B + C = 0$

$A = \frac{1}{2}$ $B = 4$ $C = -4$

general solution:

$$y = C_1 e^{-t} + C_2 \cos(t) + C_3 \sin(t) + \tfrac{1}{2} t e^{-t} + 4t - 4$$

example Form of Y only

$$y^{(4)} - y''' + y'' - y' = t^2 + 4 + t\sin(t)$$

$$r^4 - r^3 + r^2 - r = 0$$

$$r(\underbrace{r^3 - r^2 + r - 1}) = 0$$

ratio of coeff: $1, -1 \quad 1, -1$; $-1 \quad -1$ factor by grouping like last example

$$r\left[r^2(r-1) + (r-1)\right] = 0$$

$$r(r-1)(r^2+1) = 0 \qquad r=0,\ r=1,\ r=i,\ r=-i$$

$$y = C_1 + C_2 e^t + C_3 \cos(t) + C_4 \sin(t) + Y$$

right side : $t^2+4+t\sin t$

~~4/4~~

initial Y : $Y = At^2+Bt+C + (Dt+E)\cos(t) + (Ft+G)\sin(t)$

copying c_1

extra t for both

extra t

all get extra t because otherwise more copying

fixed Y: $Y = t\left(At^2+Bt+C+(Dt+E)\cos(t)+(Ft+G)\sin(t)\right)$

quick mention of 4.4 Variation of Parameters

4th-order $\quad y = u_1y_1+u_2y_2+u_3y_3+u_4y_4$

still assume $\quad u_1'y_1+u_2'y_2+u_3'y_3+u_4'y_4=0$

then

$$u_1'y_1'+u_2'y_2'+u_3'y_3'+u_4'y_4'=0$$

$$u_1'y_1''+u_2'y_2''+u_3'y_3''+u_4'y_4''=0$$

$$u_1'y_1'''+u_2'y_2'''+u_3'y_3'''+u_4'y_4'''=g(t)$$

right side

solve for u_1', u_2', u_3', u_4', then integrate

6.1 Definition of Laplace Transform

$f(t), \quad t \geq 0$

the Laplace transform of $f(t)$ is

new variable

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt}$$

Laplace transform

usually upper case of the same letter

this is one example of integral transforms (another common one is Fourier transform)

what is the Laplace transform of $f(t) = 1$?

$$\mathcal{L}\{1\} = F(s) = \int_0^\infty 1 \cdot e^{-st}\,dt$$

t is variable of integration
s is "constant" for integration

$$= \lim_{a\to\infty} \int_0^a e^{-st}\,dt = \lim_{a\to\infty} \left(-\frac{1}{s}e^{-st} \Big|_0^a\right.$$

$$= \underbrace{\lim_{a\to\infty}\left(-\frac{1}{s}e^{-sa}\right)}_{\text{want this to go to 0 so integral converges}} + \frac{1}{s} = \frac{1}{s} \qquad \boxed{\mathcal{L}\{1\} = \frac{1}{s}\,,\ s>0}$$

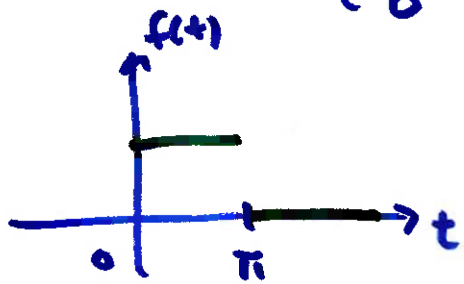
so, $s>0$

what about $f(t) = \begin{cases} 1 & 0<t<\pi \\ 0 & t \geq \pi \end{cases}$

discontinuous

BUT, piecewise continuous

(one of the requirements for $f(t)$ to have a Laplace transform)



$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\pi 1\cdot e^{-st}\,dt + \cancel{\int_\pi^\infty 0\, e^{-st}\,dt}^{\,0}$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\pi = -\frac{1}{s}e^{-\pi s} + \frac{1}{s} = \boxed{\frac{1}{s}\left(1-e^{-\pi s}\right)} \quad (s>0)$$